

Pensieve header: Improving KnotTheory`'s ColouredJones by using the built in QPochhammer / QBino-  
mial, following Stavros.

```
<< KnotTheory`
```

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Loading KnotTheory` version of February 5, 2013, 3:48:46.4762.  
Read more at http://katlas.org/wiki/KnotTheory.
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```
? ColouredJones
```

ColouredJones[K, n][q] returns the coloured Jones polynomial of a knot in colour n (i.e., in the (n+1)-dimensional representation) in the indeterminate q. Some of these polynomials have been precomputed in KnotTheory`. To force computation, use ColouredJones[K, n, Program -> "prog"][q], with "prog" replaced by one of the two available programs, "REngine" or "Braid" (including the quotes). "REngine" (default) computes the invariant for closed knots (as well as links where all components are coloured by the same integer) directly from the MorseLink presentation of the knot, while "Braid" computes the invariant via a presentation of the knot as a braid closure. "REngine" will usually be faster, but it might be better to use "Braid" when (roughly): 1) a "good" braid representative is available for the knot, and 2) the length of this braid is less than the maximum width of the MorseLink presentation of the knot.

```
{K = Knot [8, 17], n = 4}
```

```
{Knot [8, 17], 4}
```

```
{j1 = ColouredJones[K, n, Program -> "Braid"][q]} // Timing
```

KnotTheory::credits: The minimum braids representing the knots with

up to 10 crossings were provided by Thomas Gittings. See arXiv:math.GT/0401051.

$$\left\{ 3.369622, 1233 + \frac{1}{q^{40}} - \frac{3}{q^{39}} + \frac{1}{q^{38}} + \frac{5}{q^{37}} - \frac{3}{q^{36}} + \frac{1}{q^{35}} - \frac{17}{q^{34}} + \frac{6}{q^{33}} + \frac{31}{q^{32}} + \frac{1}{q^{31}} - \frac{82}{q^{29}} - \frac{16}{q^{28}} + \frac{96}{q^{27}} + \frac{69}{q^{26}} + \frac{52}{q^{25}} - \frac{216}{q^{24}} - \frac{146}{q^{23}} + \frac{120}{q^{22}} + \frac{216}{q^{21}} + \frac{260}{q^{20}} - \frac{323}{q^{19}} - \frac{393}{q^{18}} - \frac{7}{q^{17}} + \frac{340}{q^{16}} + \frac{605}{q^{15}} - \frac{292}{q^{14}} - \frac{631}{q^{13}} - \frac{265}{q^{12}} + \frac{347}{q^{11}} + \frac{945}{q^{10}} - \frac{149}{q^9} - \frac{759}{q^8} - \frac{522}{q^7} + \frac{261}{q^6} + \frac{1161}{q^5} + \frac{11}{q^4} - \frac{771}{q^3} - \frac{694}{q^2} + \frac{144}{q} + 144q - 694q^2 - 771q^3 + 11q^4 + 1161q^5 + 261q^6 - 522q^7 - 759q^8 - 149q^9 + 945q^{10} + 347q^{11} - 265q^{12} - 631q^{13} - 292q^{14} + 605q^{15} + 340q^{16} - 7q^{17} - 393q^{18} - 323q^{19} + 260q^{20} + 216q^{21} + 120q^{22} - 146q^{23} - 216q^{24} + 52q^{25} + 69q^{26} + 96q^{27} - 16q^{28} - 82q^{29} + q^{31} + 31q^{32} + 6q^{33} - 17q^{34} + q^{35} - 3q^{36} + 5q^{37} + q^{38} - 3q^{39} + q^{40} \right\}$$

```
False && {j2 = ColouredJones[K // PD, n][q]} // Timing
```

```
{0., False}
```

**j1 == j2**

$$\begin{aligned}
 & 1233 + \frac{1}{q^{40}} - \frac{3}{q^{39}} + \frac{1}{q^{38}} + \frac{5}{q^{37}} - \frac{3}{q^{36}} + \frac{1}{q^{35}} - \frac{17}{q^{34}} + \frac{6}{q^{33}} + \frac{31}{q^{32}} + \frac{1}{q^{31}} - \frac{82}{q^{29}} - \frac{16}{q^{28}} + \\
 & \frac{96}{q^{27}} + \frac{69}{q^{26}} + \frac{52}{q^{25}} - \frac{216}{q^{24}} - \frac{146}{q^{23}} + \frac{120}{q^{22}} + \frac{216}{q^{21}} + \frac{260}{q^{20}} - \frac{323}{q^{19}} - \frac{393}{q^{18}} - \frac{7}{q^{17}} + \frac{340}{q^{16}} + \frac{605}{q^{15}} - \\
 & \frac{292}{q^{14}} - \frac{631}{q^{13}} - \frac{265}{q^{12}} + \frac{347}{q^{11}} + \frac{945}{q^{10}} - \frac{149}{q^9} - \frac{759}{q^8} - \frac{522}{q^7} + \frac{261}{q^6} + \frac{1161}{q^5} + \frac{11}{q^4} - \frac{771}{q^3} - \frac{694}{q^2} + \\
 & \frac{144}{q} + 144q - 694q^2 - 771q^3 + 11q^4 + 1161q^5 + 261q^6 - 522q^7 - 759q^8 - 149q^9 + \\
 & 945q^{10} + 347q^{11} - 265q^{12} - 631q^{13} - 292q^{14} + 605q^{15} + 340q^{16} - 7q^{17} - 393q^{18} - \\
 & 323q^{19} + 260q^{20} + 216q^{21} + 120q^{22} - 146q^{23} - 216q^{24} + 52q^{25} + 69q^{26} + 96q^{27} - \\
 & 16q^{28} - 82q^{29} + q^{31} + 31q^{32} + 6q^{33} - 17q^{34} + q^{35} - 3q^{36} + 5q^{37} + q^{38} - 3q^{39} + q^{40} = j2
 \end{aligned}$$

Changes below, relative to current program:

\* Remove qExpand.

\* Remove qPochhammer[\_\_\_\_], replace by FunctionExpand[QPochhammer[\_\_\_\_]] in two places.

\* Same for qBinomial.

```
(* qQBin[n_,k_Integer,q_] /; k > 0 :=
  qQBin[n,k,q] = FunctionExpand[QPochhammer[q^(n-k+1),q,k]/QPochhammer[q,q,k]];
qQBin[n_,k_Integer,q_] /; k < 0 := qQBin[n,k,q] = 0; *)
qQBin[n_,k_,q_] := If[0 ≤ k ≤ n, QBinomial[n,k,q], 0];
```

```

Summand[K_, n_] /; Head[K] != BR := Summand[BR[K], n]
Summand[BR[s_, l_List], n_] := Module[
  {i, eqns, v, vars, sol, nulls, a = Range[s], m = s, j, B, summand}, B = Times[Times@@
    (1 /. {j_Integer /; j > 0 => Xp[a[[j]], a[[j+1]], a[[j]] = ++m, a[[j+1]] = ++m},
      j_Integer /; j < 0 => Xm[a[[-j]], a[[1-j]], a[[-j]] = ++m, a[[1-j]] = ++m})],
  Product[cl[a[[j]], j], {j, 2, s}], bt[1] * tp[a[[1]]]];
i = 0;
eqns = Flatten[
  (List@@B) /. {Xp[a_, b_, c_, d_] => {v[a] - k[+i] == v[d], v[b] + k[i] == v[c]},
    Xm[a_, b_, c_, d_] => {v[a] + k[+i] == v[d], v[b] - k[i] == v[c]},
    cl[a_, b_] => {v[a] == v[b]}, (bt | tp)[a_] => {v[a] == 0}}];
vars = v /@ (Union@@Apply[List, List@@B, {1}]);
sol = First@Solve[eqns, vars];
nulls = Union[Cases[Cases[Last/@sol,
  HoldPattern[Times[-1, _] | Plus[Times[-1, _] ..]]], _k, Infinity]];
i = 0;
summand = B /. {Xp[a_, b_, _, _] => (+i;
  q^(n/2) * q^(v[b] (k[i] - v[a])) * qQBin[v[a], k[i], 1/q] * q^(v[b] * n) *
  QPochhammer[q^(n - v[b]), 1/q, k[i]]}, Xm[a_, b_, _, _] => (+i;
  q^(-n/2) * q^(-v[a] (k[i] - v[b])) * qQBin[v[b], k[i], q] *
  q^(-v[a] * n) * QPochhammer[q^(-n + v[a]), q, k[i]]}, cl[a_, b_] =>
  q^((2 v[a] - n) / 2), (_bt) | (_tp) => 1] /. sol /. ((# -> 0) & /@ nulls);
vars = Union[Cases[summand, _k, Infinity]];
{summand /. q -> CJ`q, vars} /. Thread[Rule[vars, Array[CJ`k, Length[vars]]]]]

CJStavros[b_BR, n_Integer, opts___] :=
Module[{compute = Compute /. {opts} /. Options[ColouredJones],
  s1, vars1, s2, vars2, s, vars, rule, nv, out = 0, jj},
If[! compute, NotAvailable, {s1, vars1} = Summand[b, n];
{s2, vars2} = Summand[Mirror[b], n];
If[Length[vars1] <= Length[vars2], {vars, s} = {vars1, s1};
  rule = {CJ`q -> #}, {vars, s} = {vars2, s2}; rule = {CJ`q -> 1 / #}];
s = Together[FunctionExpand[s]] // Expand;
nv = Length[vars];
Do[out += Expand[FunctionExpand[
  s /. Thread[Rule[vars, IntegerDigits[jj, n+1, nv]]]], {jj, (n+1)^nv}];
Function@@{Expand[Simplify[out /. rule]]}]

```

**(j3 = CJStavros[K // BR, n][q]) // Timing**

$$\left\{ 10.764069, 1233 + \frac{1}{q^{40}} - \frac{3}{q^{39}} + \frac{1}{q^{38}} + \frac{5}{q^{37}} - \frac{3}{q^{36}} + \frac{1}{q^{35}} - \frac{17}{q^{34}} + \frac{6}{q^{33}} + \frac{31}{q^{32}} + \frac{1}{q^{31}} - \frac{82}{q^{29}} - \frac{16}{q^{28}} + \frac{96}{q^{27}} + \frac{69}{q^{26}} + \frac{52}{q^{25}} - \frac{216}{q^{24}} - \frac{146}{q^{23}} + \frac{120}{q^{22}} + \frac{216}{q^{21}} + \frac{260}{q^{20}} - \frac{323}{q^{19}} - \frac{393}{q^{18}} - \frac{7}{q^{17}} + \frac{340}{q^{16}} + \frac{605}{q^{15}} - \frac{292}{q^{14}} - \frac{631}{q^{13}} - \frac{265}{q^{12}} + \frac{347}{q^{11}} + \frac{945}{q^{10}} - \frac{149}{q^9} - \frac{759}{q^8} - \frac{522}{q^7} + \frac{261}{q^6} + \frac{1161}{q^5} + \frac{11}{q^4} - \frac{771}{q^3} - \frac{694}{q^2} + \frac{144}{q} + 144q - 694q^2 - 771q^3 + 11q^4 + 1161q^5 + 261q^6 - 522q^7 - 759q^8 - 149q^9 + 945q^{10} + 347q^{11} - 265q^{12} - 631q^{13} - 292q^{14} + 605q^{15} + 340q^{16} - 7q^{17} - 393q^{18} - 323q^{19} + 260q^{20} + 216q^{21} + 120q^{22} - 146q^{23} - 216q^{24} + 52q^{25} + 69q^{26} + 96q^{27} - 16q^{28} - 82q^{29} + q^{31} + 31q^{32} + 6q^{33} - 17q^{34} + q^{35} - 3q^{36} + 5q^{37} + q^{38} - 3q^{39} + q^{40} \right\}$$

**j1 == j3**

True

**(j4 = ColouredJones[K, n, Program → "Braid"][q]) // Timing**

$$\left\{ 0., 1233 + \frac{1}{q^{40}} - \frac{3}{q^{39}} + \frac{1}{q^{38}} + \frac{5}{q^{37}} - \frac{3}{q^{36}} + \frac{1}{q^{35}} - \frac{17}{q^{34}} + \frac{6}{q^{33}} + \frac{31}{q^{32}} + \frac{1}{q^{31}} - \frac{82}{q^{29}} - \frac{16}{q^{28}} + \frac{96}{q^{27}} + \frac{69}{q^{26}} + \frac{52}{q^{25}} - \frac{216}{q^{24}} - \frac{146}{q^{23}} + \frac{120}{q^{22}} + \frac{216}{q^{21}} + \frac{260}{q^{20}} - \frac{323}{q^{19}} - \frac{393}{q^{18}} - \frac{7}{q^{17}} + \frac{340}{q^{16}} + \frac{605}{q^{15}} - \frac{292}{q^{14}} - \frac{631}{q^{13}} - \frac{265}{q^{12}} + \frac{347}{q^{11}} + \frac{945}{q^{10}} - \frac{149}{q^9} - \frac{759}{q^8} - \frac{522}{q^7} + \frac{261}{q^6} + \frac{1161}{q^5} + \frac{11}{q^4} - \frac{771}{q^3} - \frac{694}{q^2} + \frac{144}{q} + 144q - 694q^2 - 771q^3 + 11q^4 + 1161q^5 + 261q^6 - 522q^7 - 759q^8 - 149q^9 + 945q^{10} + 347q^{11} - 265q^{12} - 631q^{13} - 292q^{14} + 605q^{15} + 340q^{16} - 7q^{17} - 393q^{18} - 323q^{19} + 260q^{20} + 216q^{21} + 120q^{22} - 146q^{23} - 216q^{24} + 52q^{25} + 69q^{26} + 96q^{27} - 16q^{28} - 82q^{29} + q^{31} + 31q^{32} + 6q^{33} - 17q^{34} + q^{35} - 3q^{36} + 5q^{37} + q^{38} - 3q^{39} + q^{40} \right\}$$