where δ_0 and γ_0 are the morphisms $\delta_0 : D_0 \to b_2$ and $\gamma_0 : b_1 \to E_0$.
A consequence of all of this is that the DG algorithm can be applied to a vertical complex in Ω in such a way that the others vertical complexes remain unchanged.

6. Proof of Theorem 2

The main part of the proof of Theorem 2 is to show that the composition of coherently diagonal complexes in a binary basic operator is also coherently diagonal. So, before proving this theorem, let us analyze first what occurs when in this type of operator two smoothings are embedded. Recall that $\mathcal{S}_o$ denotes the class of alternating oriented smoothings.

Proposition 6.1. *Let σ and τ be smoothings in $\mathcal{S}_o$, and let D be a suitable binary planar operator defined from a no-curl planar arc diagram with output disc D_0, input discs D_1, D_2, associated rotation constant R_D and with at least one boundary arc ending in D_1. Then there exists a closure operator C and a unary operator U such that $D(\sigma, \tau) = U(C(\sigma))$. Moreover, if (Ω, d) is C-diagonal, then $D(\Omega, \tau)$ is $(C - R(\tau) - R_D)$-diagonal.*

The last line belongs to the next proposition

coherently

a proof of the technicality instead of a proof of the main point.

Proof. To prove that the rotation constant of $D(\Omega, \tau)$ is $C - R(\tau) - R_D$, we observe that for each smoothing $\sigma\{q_\sigma\}$ in Ω the shifted rotation number satisfies $R(D(\sigma\{q_\sigma\}, \tau)) = R_D + R(\sigma\{q_\sigma\}) + R(\tau) = R_D + 2r - C + R(\tau)$. Therefore, $2r - R(D(\sigma\{q_\sigma\}, \tau)) = C - R(\tau) - R_D$ □

Proposition 6.2. *Let Ω be a coherently C-diagonal complex. Let $[\sigma_j]_j$ be a vector of degree-shifted smoothings in $\mathcal{S}_o$, all of them with the same rotation number R. Suppose that D is an appropriate binary operator defined from a no-curl planar arc diagram with associated rotation constant R_D and at least one boundary arc coming from the first input disc. Then $D(\Omega, [\sigma_j]_j)$ is a $(C - R - R_D)$-diagonal complex.*

Proof. The complex Ω is homotopy equivalent to a reduced C-diagonal complex Ω' .The complex $D(\Omega', [\sigma_j]_j)$ is the direct sum $\oplus_j [D(\Omega', \sigma_j)]$. Thus, the proposition follows from the observation that by proposition 6.1, each of its direct summands $D(\Omega', \sigma_j)$ is a coherently diagonal complex with rotation constant $C - R - R_D$. □

Lemma 6.3. *Let Ω_1 be a coherently C_1-diagonal complex. Let Ω_2 be a C_2-diagonal complex. Suppose that D is an appropriate binary operator defined from a no-curl planar arc diagram with associated rotation constant R_D and at least one boundary arc coming from the first input disc. Then $D(\Omega_1, \Omega_2)$ is $(C_1 + C_2 - R_D)$-diagonal.*

Proof. Observe that $\Omega = D(\Omega_1, \Omega_2)$ is a double complex. Indeed, if Ω_2 is the chain complex

$$\cdots \longrightarrow \Omega_2^{q-1} \longrightarrow \Omega_2^{q} \longrightarrow \Omega_2^{q+1} \cdots$$

then $\Omega_{\bullet,q}$ is the planar composition $D(\Omega_1, \Omega_2^q)$. Assume that Ω_2 is in its reduced form, then any of the smoothings in Ω_2^q has the same rotation number, $2q - C_2$. Thus, by proposition 6.2, $\Omega_{\bullet,q}$ is homotopy equivalent to a reduced diagonal complex $\Omega'_{\bullet,q}$ with rotation constant $C_1 + C_2 - 2q - R_D$. We already know that we can apply delooping and gaussian elimination in Ω involving only elements of $\Omega_{\bullet,q}$ and obtain a homotopy equivalent complex that has no changes in another vertical chain complex of Ω. In consequence, Ω is homotopy equivalent to a perturbed complex Ω' in which each $\Omega_{\bullet,q}$ has been replaced by its correspondent reduced complex $\Omega'_{\bullet,q}$. Thus, for each obtained $\Omega'_{\bullet,q}$ and each of its homological degree p, we have $2p - R(\Omega_{p,q}) = C_1 + C_2 - 2q - R_D$. Therefore, Ω' is a diagonal complex with rotation constant $C_1 + C_2 - R_D$. □