Penneys@Fields: A talk about II_1 factors

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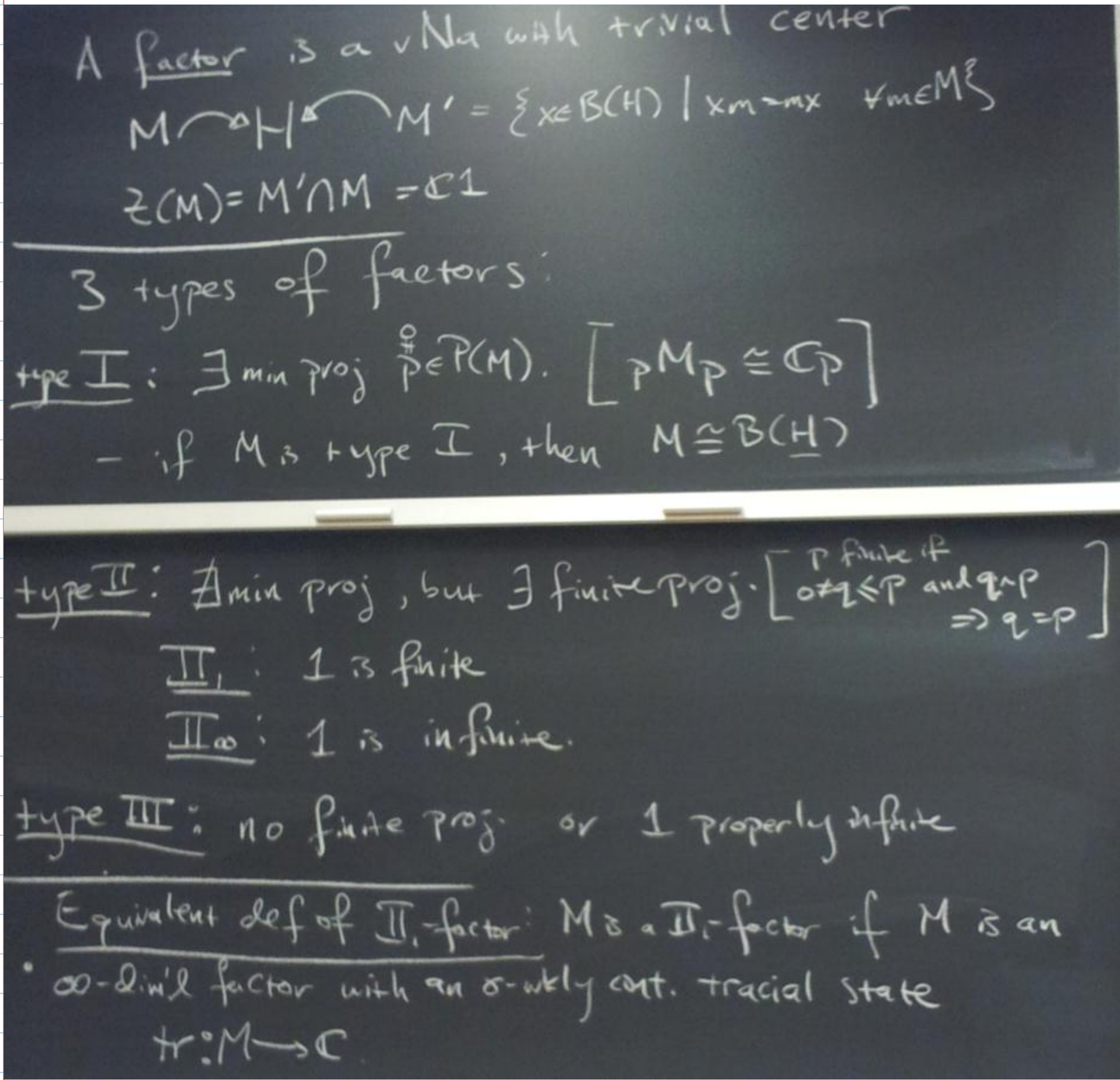
10:10 AM


A *factor* is a vNa with trivial center

$M \curvearrowright H$ $\quad M' = \{x \in B(H) \mid xm = mx \ \ \forall m \in M\}$

$Z(M) = M' \cap M = \mathbb{C}1$

3 types of factors:

type I: $\exists$ min proj $0 \neq p \in P(M)$. $[pMp \cong \mathbb{C}p]$

– if M is type I, then $M \cong B(H)$

type II: $\nexists$ min proj, but $\exists$ finite proj. [p finite if $0 \neq q \leq p$ and $q \sim p \Rightarrow q = p$]

II_1: 1 is finite

II_∞: 1 is infinite.

type III: no finite proj. or 1 properly infinite

Equivalent def of II_1-factor: M is a II_1-factor if M is an

• ∞-dim'l factor with an σ-wkly cont. tracial state $tr: M \to \mathbb{C}$

Example of a II_1 factor: Γ a countable discrete group; it acts on $\ell^2(\Gamma)$:

$$(u_g f)(h) = f(g^{-1}h) \qquad u_g \delta_h = \delta_{gh}$$

$$L\Gamma = \{u_g\}'' \subseteq B(\ell^2(\Gamma))$$

$$[u_g]_{h,k} = \langle u_g \delta_k, \delta_h \rangle = \delta_{gk,h} = \begin{cases} 1 & k = hg^{-1} \\ 0 & \text{otherwise} \end{cases}$$

u_g's are permutation matrices & they

are "generalized Toeplitz matrices"

If $x \in L\Gamma$, $x = \sum x_g u_g$

$$\left(\sum x_g u_g\right)\left(\sum y_h u_h\right) = \sum_g \left(\sum_h x_{gh^{-1}} y_h\right) u_g$$

$$\left(\sum x_g u_g\right)^* = \sum \overline{x_{g^{-1}}} u_g$$

$$tr(x) = x_e$$

$$tr(x^* x) = \sum_g |x_g|^2 < \infty \quad \text{so } (x_g) \in \ell^2(\Gamma)$$

Fact: $L\Gamma = \{x \in \ell^2(\Gamma) : x * f \in \ell^2(\Gamma) \ \forall f \in \ell^2(\Gamma)\}$

When is $L\Gamma$ a II_1 factor?

Ans: Iff all conjugacy classes of Γ except the identity are infinite.

"ICC groups" Examples: 1. $\mathbb{F}_n$ $n \geq 2$

2. S_∞: finite permutations of $\mathbb{N}$.

<u>Thm:</u> If M is a II_1 factor, then
$$tr(P(M)) = [0,1].$$

In the examples —

<u>$\mathbb{F}_n$</u>: $L\mathbb{Z} \subset L\mathbb{F}_2$, $L\mathbb{Z} \cong L^\infty(\mathbb{T}, d\theta)$

has projections of any trace.

So If u is a transposition $\frac{1 \pm u}{2}$ is a projection with trace $1/2$. By taking products & sums, get arbitrary diadic rationals.

Example $\quad R = \bigotimes^{\infty} M_{2\times 2}(\mathbb{C})$