NCGE Handout revisions

December-09-12
3:01 PM

Make a "Dyamic" NCGE calculator?
maybe next time. ...

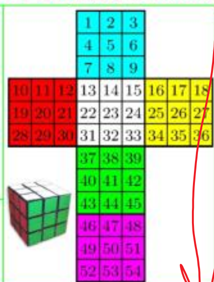
Add a reference to Knuth, http://link.springer.com/article/10.1007%2FBF01375471 + picture

Dror Bar-Natan: Talks: Mathcamp-0907:

Non-Commutative Gaussian Elimination and Rubik's Cube

Joint study with Itai Bar-Natan

The Problem. Let $G = \langle g_1, \dots, g_\alpha \rangle$ be a subgroup of S_n, with $n = O(100)$. Before you die, understand G:
1. Compute $|G|$.
2. Given $\sigma \in S_n$, decide if $\sigma \in G$.
3. Write a $\sigma \in G$ in terms of $g_1, \dots, g_\alpha$.
4. Produce *random* elements of G.



The Commutative Analog. Let $V = \mathrm{span}(v_1, \dots, v_\alpha)$ be a subspace of $\mathbb{R}^n$. Before you die, understand V.

Solution: Gaussian Elimination. Prepare an empty table,

1	2	3	4	$\cdots$	$n-1$	n

$\underbrace{}$ Space for a vector $u_4 \in V$, of the form $u_4 = (0, 0, 0, 1, *, \dots, *)$; 1 :="the pivot".

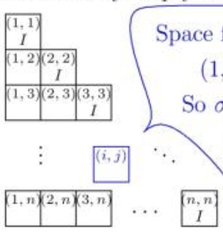
Based on an algorithm by

Otto Schreier Charles Sims

See also *Permutation Group Algorithms* by Ákos Seress.

Feed $v_1, \dots, v_\alpha$ in order. To feed a non-zero v, find its pivotal position i.
1. If box i is empty, put v there.
2. If box i is occupied, find a combination v' of v and u_i that eliminates the pivot, and feed v'.

Non-Commutative Gaussian Elimination
Prepare a mostly-empty table,



Space for a $\sigma_{i,j} \in S_n$ of the form $(1, 2, \dots, i-2, i-1, j, *, *, \dots, *)$
So $\sigma_{i,j}$ fixes $1, \dots, i-1$,
sends "the pivot" i to j and
goes wild afterwards, and
$\sigma_{i,j}^{-1}$ "does sticker j".

Feed $g_1, \dots, g_\alpha$ in order. To feed a non-identity σ, find its pivotal position i and let $j := \sigma(i)$.
1. If box (i, j) is empty, put σ there.
2. If box (i, j) contains $\sigma_{i,j}$, feed $\sigma' := \sigma_{i,j}^{-1}\sigma$.

The Twist. When done, for every occupied (i, j) and (k, l), feed $\sigma_{i,j}\sigma_{k,l}$. Repeat until the table stops changing.

Claim. The process stops in our lifetimes, after at most $O(n^6)$ operations. Call the resulting table T.

Claim. Anything fed in T is a monotone product in T:

f was fed $\Rightarrow f \in M_1 := \{\sigma_{1,j_1}\sigma_{2,j_2}\cdots\sigma_{n,j_n} : \forall i, j_i \geq i \;\&\; \sigma_{i,j_i} \in T\}$

The Generators

```
In[1]:= gs = {
  purple = P[18,27,36,4,5,6,7,8,9,3,11,12,13,14,15,16,17,
   45,2,20,21,22,23,24,25,26,44,1,29,30,31,32,33,34,35,43,
   37,38,39,40,41,42,10,19,28,52,49,46,53,50,47,54,51,48],
  white = P[1,2,3,4,5,6,16,25,34,10,11,9,15,24,33,39,17,
   18,19,20,8,14,23,32,38,26,27,28,29,7,13,22,31,37,35,36,
   12,21,30,40,41,42,43,44,45,46,47,48,49,50,51,52,53,54],
  green = P[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,
   19,20,21,22,23,24,25,26,27,31,32,33,34,35,36,48,47,46,
   39,42,45,38,41,44,37,40,43,30,29,28,49,50,51,52,53,54],
  blue = P[3,6,9,2,5,8,1,4,7,54,53,52,10,11,12,13,14,15,
   19,20,21,22,23,24,25,26,27,28,29,30,31,32,33,34,35,36,
   37,38,39,40,41,42,43,44,45,46,47,48,49,50,51,18,17,16],
  red = P[13,2,3,22,5,6,31,8,9,12,21,30,37,14,15,16,17,
   18,11,20,29,40,23,24,25,26,27,10,19,28,43,32,33,34,35,
   36,46,38,39,49,41,42,52,44,45,1,47,48,4,50,51,7,53,54],
  yellow = P[1,2,48,4,5,51,7,8,54,10,11,12,13,14,3,18,27,
   36,19,20,21,22,23,6,17,26,35,28,29,30,31,32,9,16,25,34,
   37,38,15,40,41,24,43,44,33,46,47,39,49,50,42,52,53,45]
};   Enter
```

Theorem. $G = M_1$. *G^{-1} is more fun!*

$G = M_1 := \{\sigma_{1,j_1}\sigma_{2,j_2}\cdots\sigma_{n,j_n} : \forall i, j_i \geq i \text{ and } \sigma_{i,j_i} \in T\}.$

Proof. The inclusions $M_1 \subset G$ and $\{g_1, \dots, g_\alpha\} \subset M_1$ are obvious. The rest follows from the following

Lemma. M_1 is closed under multiplication.

Proof. By backwards induction. Let

$$M_k := \{\sigma_{k,j_k}\cdots\sigma_{n,j_n} : \forall i \geq k, j_i \geq i \text{ and } \sigma_{i,j_i} \in T\}.$$

Clearly $M_n M_n \subset M_n$. Now assume that $M_5 M_5 \subset M_5$ and show that $M_4 M_4 \subset M_4$. Start with $\sigma_{8,j} M_4 \subset M_4$:

$$\sigma_{8,j}(\sigma_{4,j_4}M_5) \overset{1}{=} (\sigma_{8,j}\sigma_{4,j_4})M_5 \overset{2}{\subset} M_4 M_5$$

$$\overset{3}{=} \sigma_{4,j_4}(M_5 M_5) \overset{4}{\subset} \sigma_{4,j_4} M_5 \subset M_4$$

(1: associativity, 2: thank the twist, 3: associativity and tracing i_4, 4: induction). Now the general case

$$(\sigma_{4,j_4'}\sigma_{5,j_5'}\cdots)(\sigma_{4,j_4}\sigma_{5,j_5}\cdots)$$

falls like a chain of dominos.

Problem Solved!

A Demo Program

```
In[2]:= ($RecursionLimit = 2^16;
  n = 54;
  P /: p_P ** P[a___] := p[[{a}]];
  Inv[p_P] := P @@ Ordering[p];
  Feed[P @@ Range[n]] := Null;
  Feed[p_P] := Module[{i, j},
    For[i = 1, p[[i]] == i, ++i];
    j = p[[i]];
    If[Head[s[i, j]] === P,
      Feed[Inv[s[i, j]] ** p],
    (* Else *) s[i, j] = p;
      Do[If[Head[s[k, l]] == P,
        Feed[s[i, j] ** s[k, l]];
        Feed[s[k, l] ** s[i, j]]
      ], {k, n}, {l, n}]
    ]]);   Enter
```

www.geocities.com/jaapsch/puzzles/

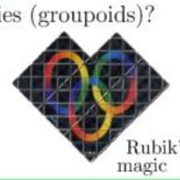
www.powerstrike.net/puzzles/

http://lodelnic.free.fr/

Homework Problem 1. Can you do cosets?

Homework Problem 2. Can you do categories (groupoids)?

7	9	2	5
1	4	8	3
6	10	11	12
13	14	15	



Rubik's magic

The Results

that's cool!

Modernize.

From 11-1100/Hour 1-2:

Claim 1 Every σ_{ij} in T is in G.

Claim 2 Anything fed to T is now a monotone product $\sigma_{1j_1}\sigma_{2j_2}\sigma_{3j_3}\dots$ $\quad j_i \geq i$

Claim 3 If two monotone products are equal,

$$\sigma_{1j_1}\dots\sigma_{nj_n} = \sigma_{1j_1'}\dots\sigma_{nj_n'}$$

then all the indices are equal, $\forall i\ j_i = j_i'$.

Claim 4 Let $M_k = \left\{\begin{matrix}\text{monotone products}\\ \text{beginning with } k\end{matrix}\right\} = \{\sigma_{kj_k}\cdot\dots\sigma_{nj_n}\}$, then for every k, $M_k \cdot M_k \subset M_k$ (and so each M_k is a subgroup of S_n.

Proof Clearly $M_nM_n \subset M_n$. Now assume that $M_5M_5 \subset M_5$ and show that $M_4M_4 \subset M_4$. Start with $\sigma_{8,j}M_4 \subset M_4$:

$$\sigma_{8,j}(\sigma_{4,j_4}M_5) \overset{1}{=} (\sigma_{8,j}\sigma_{4,j_4})M_5 \overset{2}{\subset} M_4M_5$$

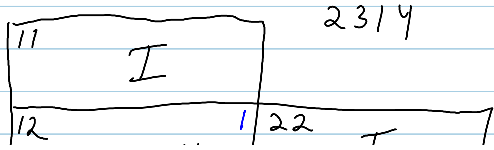
$$\overset{3}{=} \sigma_{4,j_4}(M_5M_5) \overset{4}{\subset} \sigma_{4,j_4}M_5 \subset M_4$$

Claim 5 $M_1 = G$ and we have achieved all of our goals [except there is a hidden problem].

→ then do goals 1, 2, 3, 4 and the 0: "in our lifetime".

done line

Example $\sigma_1 = (123)$ $\sigma_2 = (12)(34)$, in S_4
$\qquad\quad\ \ = 2314 \qquad\quad\ = 2143$



I			
12 $\sigma_1 = 2314$ 1	22 I		
13 $\sigma_{12}^2 = 3124$ 2	23 $\sigma_{12}^{-1}\sigma_2 = 1342$ 3	33 I	
14 $\sigma_{23}\sigma_{13} = 4132$ 5	24 $\sigma_{13}^{-1}\sigma_{23}\sigma_{12} = 1423$ 4	34	44 I

Feed $\sigma_1 = 2314$ ... Fed @ σ_{12}

Feed $\sigma_{12}^2 = 3124$ ... Fed @ σ_{13}

Feed $\sigma_2 = 2143$ ... Feed $\sigma_{12}^{-1}\sigma_2 = 1342$ .. Fed @ σ_{23}

Feed $\sigma_{12}\sigma_{23} = 2143$ ... Feed $\sigma_{12}^{-1}\sigma_{12}\sigma_{23} = \sigma_{23}$ ....

No point feeding $\sigma_{ij}\sigma_{kl}$ if $i<k$?

Feed $\sigma_{23}\sigma_{12} = 3412$ ... Feed $\sigma_{13}^{-1}\sigma_{23}\sigma_{12} = 1423$ ... to σ_{24}

Feed $\sigma_{23}\sigma_{13} = 4132$ .. to σ_{14}

Feed $\sigma_{24}\sigma_{12} = 4213$ ... Feed $\sigma_{14}^{-1}\sigma_{24}\sigma_{12} = 1423$ ... drop.

$\Longrightarrow |G| = 4\cdot 3\cdot 1\cdot 1 = 12.$ Is $4123 \in G$?

Write 2431 in terms of $\sigma_{1,2}$.

Non Commutative Gaussian Elimination @ MAT 1100

By Dror Bar-Natan

Amended from a similar notebook by Dror Bar-Natan and Itai Bar-Natan. The original version is at http://www.math.toronto.edu/~drorbn/Misc/SchreierSimsRubik/.

Pensieve Header: Non Commutative Gaussian Ellimination @ MAT 1100 - as on handout + a printout of the filling table. See more at pensieve://2009-07/.

Program 0

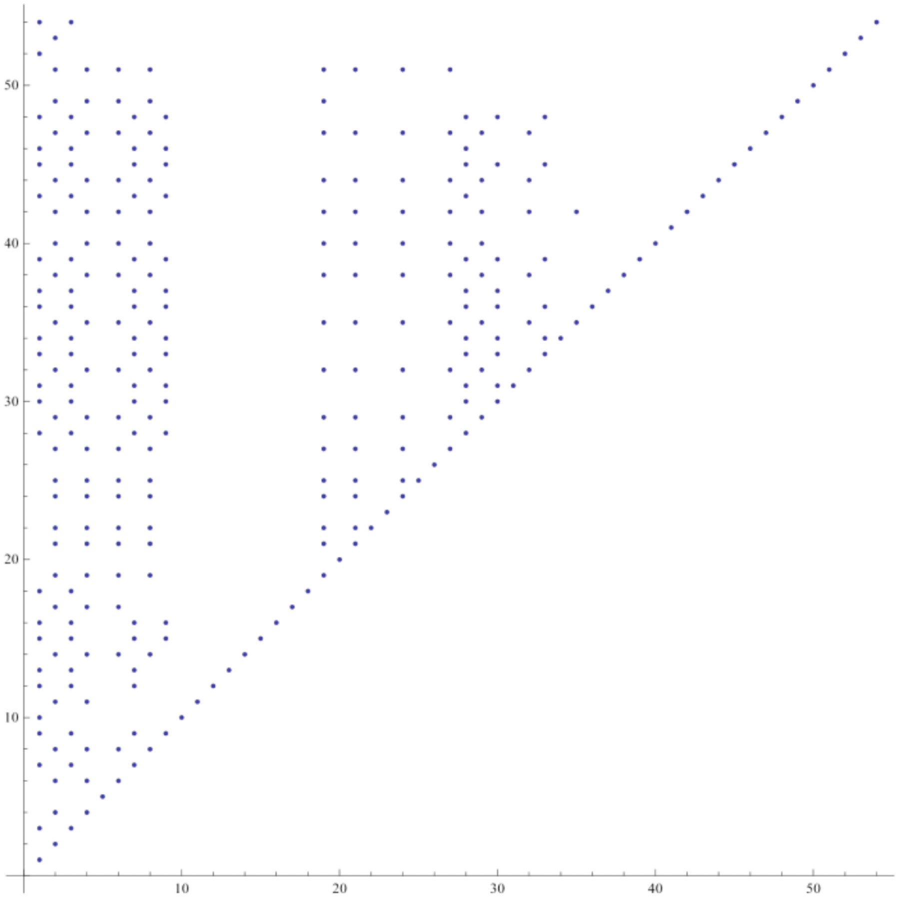
```
gs = {purple = P[18, 27, 36, 4, 5, 6, 7, 8, 9, 3, 11, 12, 13, 14,
      15, 16, 17, 45, 2, 20, 21, 22, 23, 24, 25, 26, 44, 1, 29, 30, 31, 32, 33, 34,
      35, 43, 37, 38, 39, 40, 41, 42, 10, 19, 28, 52, 49, 46, 53, 50, 47, 54, 51, 48],
    white = P[1, 2, 3, 4, 5, 6, 16, 25, 34, 10, 11, 9, 15, 24, 33, 39, 17, 18, 19,
      20, 8, 14, 23, 32, 38, 26, 27, 28, 29, 7, 13, 22, 31, 37, 35, 36, 12,
      21, 30, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54],
    green = P[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20,
      21, 22, 23, 24, 25, 26, 27, 31, 32, 33, 34, 35, 36, 48, 47, 46, 39,
      42, 45, 38, 41, 44, 37, 40, 43, 30, 29, 28, 49, 50, 51, 52, 53, 54],
    blue = P[3, 6, 9, 2, 5, 8, 1, 4, 7, 54, 53, 52, 10, 11, 12, 13, 14, 15, 19, 20,
      21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37,
      38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 18, 17, 16],
    red = P[13, 2, 3, 22, 5, 6, 31, 8, 9, 12, 21, 30, 37, 14, 15, 16, 17, 18, 11, 20,
      29, 40, 23, 24, 25, 26, 27, 10, 19, 28, 43, 32, 33, 34, 35, 36, 46,
      38, 39, 49, 41, 42, 52, 44, 45, 1, 47, 48, 4, 50, 51, 7, 53, 54],
    yellow = P[1, 2, 48, 4, 5, 51, 7, 8, 54, 10, 11, 12, 13, 14, 3, 18, 27, 36, 19,
      20, 21, 22, 23, 6, 17, 26, 35, 28, 29, 30, 31, 32, 9, 16, 25, 34, 37,
      38, 15, 40, 41, 24, 43, 44, 33, 46, 47, 39, 49, 50, 42, 52, 53, 45]};

($RecursionLimit = 2^16;
  n = 54;
  P /: p_P ** P[a___] := p[[{a}]];
  Inv[p_P] := P @@ Ordering[p];
  Feed[P @@ Range[n]] := Null;
  Feed[p_P] := Module[{i, j},
    For[i = 1, p[[i]] == i, ++i]; j = p[[i]];
    If[Head[s[i, j]] === P,
     Feed[Inv[s[i, j]] ** p],
     (*Else*) s[i, j] = p;
     Do[If[Head[s[k, l]] == P,
        Feed[s[i, j] ** s[k, l]];
        Feed[s[k, l] ** s[i, j]]
       ],
      {k, n}, {l, n}]
    ]]
 );

(Feed[#]; Product[1 + Length[Select[Range[n], Head[s[i, #]] === P &]], {i, n}]) & /@ gs

{4, 16, 159 993 501 696 000, 21 119 142 223 872 000, 43 252 003 274 489 856 000, 43 252 003 274 489 856 000}
```

```
Images[i_] := {i}~Join~Select[Range[n], Head[s[i, #]] === P &];
ListPlot[
 Join @@ Table[{i, #} & /@ Images[i], {i, n}],
 AspectRatio → 1
]
```




```
43 252 003 274 489 856 000 / (8! * 3^8 * 12! * 2^12)
```

$\frac{1}{12}$