Dror Bar-Natan: Statement of Recent Work (September 2012)

Abstract Preliminaries. A *filtered vector space* is a vector space $V = V_0$ (usually infinite dimensional, usually one with elements whose study is hard yet desirable) along with a decreasing sequence of subspaces $V = V_0 \supset V_1 \supset V_2 \supset \dots$. I like to think of V as if it contains information that could be studied inductively, and of V_n as "that part of V that is irrelevant until day n of the study". Thus the quotient V/V_n is what we could or should have studied before day n. On day n we may study V_n, though we still don't care about V_{n+1}, so the quotient V_n/V_{n+1} is precisely what we need to study on that day. Thus the direct sum $\operatorname{gr} V := \bigoplus_n V_n/V_{n+1}$ (also known as "the *associated graded space of* V") can be viewed as "V, sliced out for an inductive study". An *expansion* for V is a map $Z : V \to \operatorname{gr} V$ satisfying a simple non-degeneracy condition. An expansion may be thought of as a machine that breaks any element of V into a sequence of easier parts, with part n ready for study on day n. Often times the spaces V_n/V_{n+1} are much simpler than V, and so there is great interest in finding "good" expansions.

What's "good"? Often times V will come with various algebraic operations — various composition laws, or various other ways to transform its elements. Under mild conditions, these operations will themselves be amenable to inductive study — meaning that they induce similar operations on $\operatorname{gr} V$. A *homomorphic* ("good") expansion is an expansion $Z : V \to \operatorname{gr} V$ that intertwines the operations of V with the operations of $\operatorname{gr} V$. A homomorphic expansion allows for an even better inductive probing of V — not only the elements of V can be studied in the simplified inductive context, but so can the relations between the elements of V and the operations applied to these elements.

Generally speaking, mere expansions are cheap and easy to get. Homomorphic expansions, on the other hand, are expensive and valuable. They don't always exist, when they exist they are often hard to construct, yet when they are available, they are often very useful.

Knots. The set $\mathcal{K}$ of knots ("**topology**", below), for example, can be made into a vector space by allowing formal linear combinations and can then be filtered using the "Vassiliev filtration" which will not be recalled here except by its essence, the formula "⤫ → ⤫ − ⤫" (I was amongst the earliest contributors; see [BN1]). The resulting "$\operatorname{gr} V$" is the space $\mathcal{A}$ of chord diagrams ("**combinatorics**", below). Slightly generalizing to "parenthesized tangles", the construction of a homomorphic expansion Z turns out to depend mostly on the choice of one very special element $\Phi = Z(|/|)$ of $\mathcal{A}$, which has to satisfy some complicated equations whose origin is in category theory — mostly the "pentagon" and "hexagon" equations ("**high algebra**", below; I had a role in that too — see [BN2, BN3]). Further, it turns out that the space $\mathcal{A}$ can be re-interpreted as a space of formulas that make sense in any appropriate Lie algebra ("**low algebra**", below, [BN1]), and hence much that is done with and about Z and Φ has a Lie-theoretic and representation-theoretic meaning. (A lovely example is the explanation of the Lie-theoretic Duflo-isomorphism as the knot theoretic "$1 + 1 = 2$" [BLT]).

u, v, and w. Renaming "knots" to be "u-knots" ("u" for usual), I found that the above pattern of **topology** leading to **combinatorics** by means of a natural filtration, leading to **low algebra** by interpreting the combinatorics as the combinatorics of formulas, and to **high algebra** by the study of expansions, persists for several other classes of knots. A quick summary is in the table below:

	u-Knots	v-Knots	w-Knots
Topology	Ordinary (usual) knotted objects in 3D — braids, knots, links, tangles, knotted graphs, etc.	Virtual knotted objects — "algebraic" knotted objects, or "not specifically embedded" knotted objects; knots drawn on a surface, modulo stabilization.	Ribbon knotted objects in 4D; "flying rings". Like v, but also with "overcrossings commute".
Combinatorics	Chord diagrams and Jacobi diagrams, modulo $4T$, STU, IHX, etc.	Arrow diagrams and v-Jacobi diagrams, modulo $6T$ and various "directed" STUs and IHXs, etc.	Like v, but also with "tails commute". Only "two in one out" internal vertices.
Low Algebra	Finite dimensional metrized Lie algebras, representations, and associated spaces.	Finite dimensional Lie bi-algebras, representations, and associated spaces.	Finite dimensional co-commutative Lie bi-algebras (i.e., $\mathfrak{g} \ltimes \mathfrak{g}^*$), representations, and associated spaces.
High Algebra	The Drinfel'd theory of associators.	Likely, quantum groups and the Etingof-Kazhdan theory of quantization of Lie bi-algebras.	The Kashiwara-Vergne-Alekseev-Torossian theory of convolutions on Lie groups and Lie algebras.

A more complete version of the above table would contain a few further rows, for quantum field theory, for configuration space integrals, for graph homology, and perhaps more. It may also contain some further columns (deformation quantization of Poisson structures "p" has entries in all rows except the topology row. I much want to know what "p-topology" would be). Also, the table fails to indicate that there are maps between the entities in the topology row — specifically, there are maps $u \to v \to w$. These maps have analogs or implications in all other rows, serving to explain the otherwise mysterious connections that exists between, say, Drinfel'd associators and solutions of the Kashiwara-Vergne problem [AT, BD3].

Thus my most significant scientific work over the last 5 years had been the assembly of the above table. Much of it is still unwritten. The written parts include:

- My paper and series of videos [BD3] (with Dancso) contain the complete "w-story" from the topology of 2-dimensional knots in 4-dimensional space to the high algebra of Kashiwara-Vergne and Alekseev-Torossian and its relationship with Drinfel'd associators.
- [BD2] is about the u-column, describing the relationship between knotted trivalent graphs and associators.
- [BD1] lives at u cross high algebra, explaining (following Furusho) why under certain conditions, the pentagon equation implies the hexagon equation.
- [BHLR] contains strong computational evidence that some 18 variants of the combinatorics entry of the v column are "correct".

Many of the not-yet-written parts are available online as videotaped lectures accompanied by detailed summary handouts. With ω :=`http://www.math.toronto.edu/~drorbn/Talks/`, these include ω/Aarhus-0706 on potential applications to topology, ω/Trieste-0905, ω/Goettingen-1004, and ω/Caen-1206 on the whole table, ω/MSRI-0808, ω/Bonn-0908, and ω/Montpellier-1006 on the w column, ω/Toronto-11011 on the relationship between u and the Grothendieck-Teichmuller group, ω/Tennessee-1103 on the "configuration spaces" row that was omitted above, ω/SwissKnots-1105 on my hopes for the v column, ω/Sandbjerg-0810 on a "penultimate" Alexander invariant, ω/Chicago-1009 on the relationship between w and especially the "$ax+b$" Lie algebra and the Alexander polynomial, ω/Regina-1206 on an Alexander polynomial offshoot of the w story, which appears to be an "ultimate" Alexander invariant, and ω/Hamburg-1208 on a non-commutative generalization of the Alexander polynomial and its relationship with the BF quantum field theory.

References

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[BD3] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects: From Alexander to Kashiwara and Vergne,* in preparation, see `http://drorbn.net/index.php?title=WKO`.

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