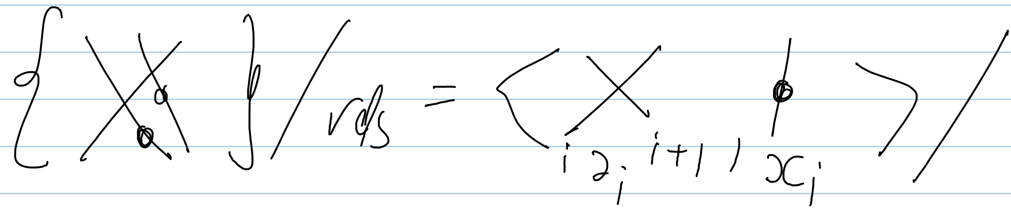
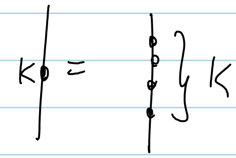
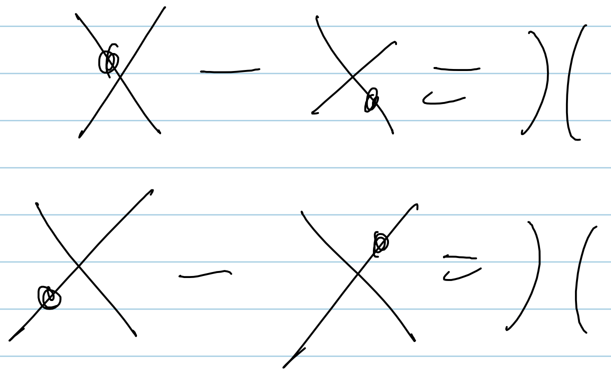
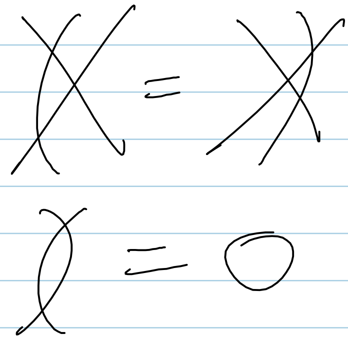
Khovanov: NilHecke algebra and its friends

March-16-12
11:11 AM

$NH_n =$ 

faraway generators commute, 





Can move all dots above all crossings.

Flag Variety

$$X = \{ 0 \subset L_1 \subset L_2 \subset \ldots \subset L_{n-1} \subset \mathbb{C}^n \mid \dim L_k = k \}$$

$$H^*(X, k) = K[x_1 \ldots x_n] \Big/ \begin{matrix}\text{symmetric poly.} \\ \text{w/o constant terms.}\end{matrix}$$

$X_k =$ partial flags $=$

$$= \{ 0 \subset L_1 \ldots \subset \widehat{L_k} \subset \ldots \}$$

Have fibration

$$\begin{array}{c} X \\ \big\downarrow \mathbb{P}^1 \\ X_k \end{array}$$

$$\partial_k : H^*(X_k) \longrightarrow H^*(X) \longrightarrow H_*(X) \longrightarrow H^*(X_k) \longrightarrow H^*(X_k)$$

○ by NH_n. This rep is faithful if one passes to GL_n-equivariant cohomology.

∂_k acts on $k[x_1 \dots x_n]$ by

$$\partial_k f = \frac{f - f(\dots x_{k+1}, x_k \dots)}{x_k - x_{k+1}}$$

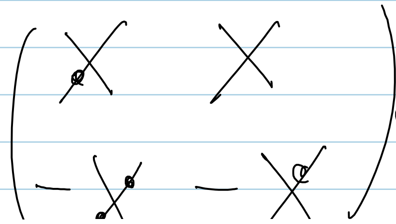
This is faithful & commutes w/ the action of symmetric polynomials:

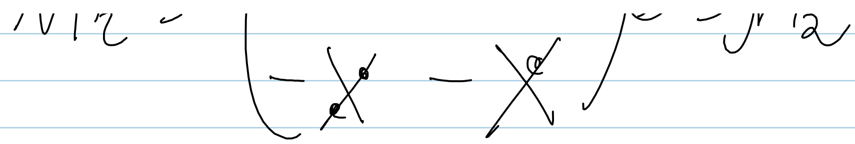
$$NH_n \xrightarrow{\sim} End_{Sym_n}(k[x_1 \dots x_n])$$

$$\cong Mat(n!, Sym_n)$$

(Taking care of grading, dim $= [n]!$)

Example

$NH_2 \cong$  $\otimes\, Sym_2$



NH_n categorifies one half of quantum $sl(2)$:

$$\left\langle E, F, K : \begin{matrix} KE = q^2 EK \\ KF = q^{-2} FK \end{matrix} \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}} \right\rangle$$

$U^+ = \langle E \rangle$ set $E^{(n)} = \dfrac{E^n}{[n]!}$

w/ $[n] = \dfrac{q^n - q^{-n}}{q - q^{-1}}$

$$\boxed{E^{(n)} E^{(m)} = \begin{bmatrix} n+m \\ n \end{bmatrix} E^{(n+m)}}$$

NH_n categorifies U^+:

$K_0(NH_n) \cong K_0(sym_n) = \mathbb{Z}[q, q^{-1}]$

w/ $[NH_n] = [n!]\,[sym_n]$

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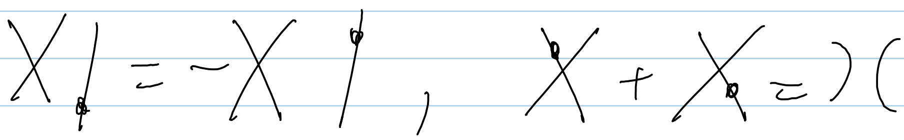
There is an odd version of all this!



etc.