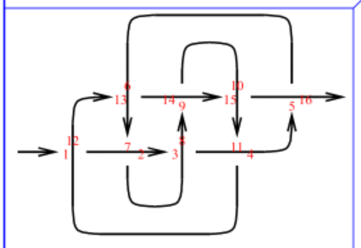
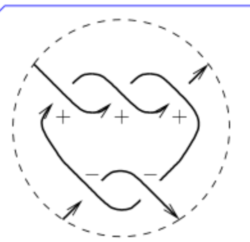
Meta–Groups, Meta–Bicrossed–Products, and the Alexander Polynomial, 1

Dror Bar–Natan at Knots in Washington XXXIV
http://www.math.toronto.edu/~drorbn/Talks/GWU-1203/

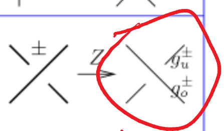
Abstract. A straightforward proposal for a group-theoretic invariant of knots fails if one really means groups, but works once generalized to meta-groups (to be defined). We will construct one complicated but elementary meta-group as a meta-bicrossed-product (to be defined), and explain how the resulting invariant is a not-yet-understood generalization of the Alexander polynomial, while at the same time being a specialization of a somewhat-understood "universal finite type invariant of w-knots" and of an elusive "universal finite type invariant of v-knots".

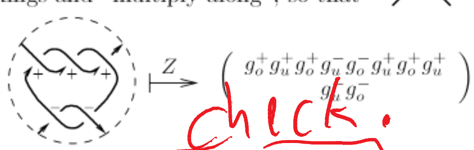




"divide and conquer"

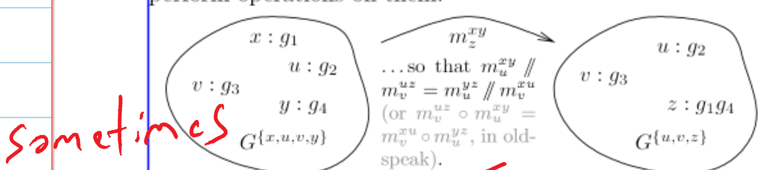
Idea. Given a group G and two "YB" pairs $R^\pm = (g_o^\pm, g_u^\pm) \in G^2$, map them to xings and "multiply along", so that





This Fails! R2 implies that $g_o^\pm g_u^\mp = e$ and then R3 implies that g_o^+ and g_u^+ commute, so the result is a simple counting invariant.

A Group Computer. Given G, can store group elements and perform operations on them:



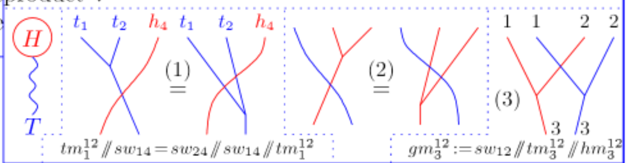
Also has S_x for inversion, e_x for unit insertion, d_x for register deletion, Δ_{xy}^z for element cloning, ρ_y^x for renamings, and $(D_1, D_2) \mapsto D_1 \cup D_2$ for merging, and many obvious composition axioms relating those. $P = \{x : g_1, y : g_2\} \Rightarrow P = \{d_y P\} \cup \{d_x P\}$

A Meta-Group. Is a similar "computer", only its internal structure is unknown to us. Namely it is a collection of sets $\{G_X\}$ indexed by all finite sets X, and a collection of operations m_z^{xy}, S_x, e_x, d_x, Δ_{xy}^z, ρ_y^x, and $\cup$, satisfying the exact same *linear* properties.

Example 1. The non-meta example, $G_X := G^X$.

Example 2. $G_X := M_{X\times X}(\mathbb{Z})$, with simultaneous row and column operations, and "block diagonal" merges.

Bicrossed Products. If $G = HT$ is a group presented as a product of two of its subgroups, with $H \cap T = \{e\}$, then also $G = TH$ and G is determined by H, T, and the "swap" map $sw^{th} : (t, h) \mapsto (h', t')$ defined by $th = h't'$. The map sw satisfies (1) and (2) below; conversely, if $sw : T \times H \to H \times T$ satisfies (1) and (2) (+ lesser conditions), then (3) defines a group structure on $H \times T$, the "bicrossed product".



A **Meta-Bicrossed-Product** is a collection of sets $\beta(H,T)$ and operations tm_z^{xy}, hm_z^{xy} and sw_{xy}^{th} (and lesser ones), such that tm and hm are "associative" and (1) and (2) hold (+ lesser conditions). A meta-bicrossed-product defines a meta-group with $G_X := \beta(X,X)$ and gm as in (3).

β Calculus. Let $\beta(H,T)$ be

$$\left\{ \begin{array}{c|ccc} \omega & h_1 & h_2 & \cdots \\ \hline t_1 & \alpha_{11} & \alpha_{12} & \cdot \\ t_2 & \alpha_{21} & \alpha_{22} & \cdot \\ \vdots & \cdot & \cdot & \cdot \end{array} \middle| \begin{array}{l} h_j \in H,\ t_i \in T, \text{ and } \omega \text{ and} \\ \text{the } \alpha_{ij} \text{ are Laurent poly-} \\ \text{nomials in variables } T_i, \text{ in} \\ \text{bijection with the } t_i\text{'s} \end{array} \right\},$$

$$tm_z^{xy} : \begin{array}{c|c} \omega & \cdots \\ \hline t_x & \alpha \\ t_y & \beta \\ \vdots & \gamma \end{array} \mapsto \begin{array}{c|c} \omega & \cdots \\ \hline t_z & \alpha+\beta \\ \vdots & \gamma \end{array}, \qquad \begin{array}{c|c} \omega_1 & H_1 \\ \hline T_1 & \alpha_1 \end{array} \cup \begin{array}{c|c} \omega_2 & H_2 \\ \hline T_2 & \alpha_2 \end{array} = \begin{array}{c|cc} \omega_1\omega_2 & H_1 & H_2 \\ \hline T_1 & \alpha_1 & 0 \\ T_2 & 0 & \alpha_2 \end{array},$$

$$hm_z^{xy} : \begin{array}{c|ccc} \omega & h_x & h_y & \cdots \\ \hline \vdots & \alpha & \beta & \gamma \end{array} \mapsto \begin{array}{c|cc} \omega & h_z & \cdots \\ \hline \vdots & \alpha+\beta+\langle\alpha\rangle\beta & \gamma \end{array},$$

$$sw_{xy}^{th} : \begin{array}{c|cc} \omega & h_y & \cdots \\ \hline t_x & \alpha & \beta \\ \vdots & \gamma & \delta \end{array} \mapsto \begin{array}{c|cc} \omega\epsilon & h_y & \cdots \\ \hline t_x & \alpha(1+\langle\gamma\rangle/\epsilon) & \beta(1+\langle\gamma\rangle/\epsilon) \\ \vdots & \gamma/\epsilon & \delta-\gamma\beta/\epsilon \end{array},$$

where $\epsilon := 1+\alpha$, $\langle\alpha\rangle := \sum_i \alpha_i$, and $\langle\gamma\rangle := \sum_{i\neq x}\gamma_i$, and let

$$R_{xy}^p := \begin{array}{c|cc} 1 & h_x & h_y \\ \hline t_x & 0 & T_x - 1 \\ t_y & 0 & 0 \end{array} \qquad R_{xy}^m := \begin{array}{c|cc} 1 & h_x & h_y \\ \hline t_x & 0 & T_x^{-1} - 1 \\ t_y & 0 & 0 \end{array}.$$

Theorem. Z^β is a tangle invariant (and even more). Restricted to knots, the ω part is the Alexander polynomial. Restricted to links, it contains the multivariable Alexander polynomial. Restricted to braids, it is equivalent to the Burau representation.

Why Happy? • Applications to w-knots. • Everything that I know about the Alexander polynomial can be expressed cleanly in this language (even if without proof), except HF, but including genus, ribbonness, cabling, v-knots, knotted graphs, etc., and there's potential for vast generalizations. • Fits on one sheet, including implementation.