Pensieve header: Perturbative β-calculations.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\2012-03"];
<< betaCalculus.m
```

```
Clear[ħ];
$PerturbativeDegree = 3;
βSimplify[expr_] := Replace[
   Series[Normal[expr], {ħ, 0, $PerturbativeDegree}],
   sd_SeriesData :> MapAt[Expand, sd, 3]
  ];
βCollect[B[ω_, μ_]] := B[
   βSimplify[ω],
   βSimplify[μ]
  ];
```

The Knot-Theoretic Equations

```
{
  V0 = βCollect[
     B[ω[ħ c1, ħ c2], α[ħ c1, ħ c2] t[1] h[1] +
       β[ħ c1, ħ c2] t[1] h[2] + γ[ħ c1, ħ c2] t[2] h[1] + δ[ħ c1, ħ c2] t[2] h[2]]
    ] /. {
     (ϵ : (α | β | γ | δ | ω | κ))[___] :> ϵ0,
     (ϵ : (α | β | γ | δ | ω | κ))^(k___)[___] :> ϵFromDigits[{k}]
    },
  C0 = βCollect[B[κ[ħ c1], 0]] /. {
     (ϵ : (α | β | γ | δ | ω | κ))[___] :> ϵ0,
     (ϵ : (α | β | γ | δ | ω | κ))^(k___)[___] :> ϵFromDigits[{k}]
    },
  eqns1 = HardR4[V0],
  eqns2 = TwistEq[V0],
  eqns3 = And[(V0 // dη[1]) == B[1, 0], (V0 // dη[2]) == B[1, 0]],
  eqns4 = V0 ** (V0 // dA[1] // dA[2]) == B[1, 0],
  eqns5 = CapEquation[V0, C0],
  eqns6 = (C0 // tη[1]) == B[1, 0],
  eqns7 = (V0 == Rot120[V0])
 } // ColumnForm
```

A very large output was generated. Here is a sample of it:

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```
eqns = eqns1 && eqns2 && eqns3 && eqns4 && eqns5 && eqns6 && (β0 == -1 / 4);
```

```
sol = SolveAlways[eqns, {ℏ, c1, c2}]
```

$$\left\{\left\{\alpha_{30}\to 0,\ \beta_{30}\to \frac{1}{64}\left(-1+48\beta_1+64\gamma_3-96\gamma_{20}\right),\right.\right.$$
$$\beta_{21}\to \frac{1}{192}\left(-1-48\beta_1-192\beta_{11}+192\gamma_{12}-96\gamma_{20}\right),$$
$$\beta_{12}\to \frac{1}{192}\left(-1-48\beta_1-192\beta_{11}-96\gamma_{20}+192\gamma_{21}\right),$$
$$\beta_3\to \frac{1}{64}\left(-1+48\beta_1-96\gamma_{20}+64\gamma_{30}\right),\ \delta_3\to 0,\ \alpha_{21}\to \frac{1}{192}\left(-1+48\beta_1-96\gamma_{20}+192\delta_{12}\right),$$
$$\alpha_{12}\to \frac{1}{32}\left(-8\beta_1-32\beta_{11}-8\delta_{10}-16\delta_{20}+32\delta_{21}+3\kappa_1-96\delta_{10}\kappa_1-32\kappa_1^3+32\kappa_3\right),$$
$$\alpha_3\to \frac{1}{64}\left(-1-96\gamma_{20}+48\delta_{10}+96\delta_{20}+64\delta_{30}\right),\ \gamma_{11}\to\beta_1+\beta_{11},$$
$$\omega_{12}\to \frac{1}{32}\left(3\kappa_1-96\delta_{10}\kappa_1-32\kappa_1^3+32\kappa_3\right),\ \omega_{21}\to \frac{1}{32}\left(3\kappa_1-96\delta_{10}\kappa_1-32\kappa_1^3+32\kappa_3\right),$$
$$\omega_3\to 0,\ \omega_{30}\to 0,\ \beta_{20}\to -\beta_1+\gamma_{20},\ \beta_2\to \frac{1}{48}\left(1-48\beta_1+48\gamma_{20}\right),\ \gamma_2\to \frac{1}{48}\left(1+48\gamma_{20}\right),\ \alpha_{20}\to 0,$$
$$\alpha_{11}\to \frac{1}{192}\left(1-48\delta_{10}-96\delta_{20}+18\kappa_1-576\delta_{10}\kappa_1-192\kappa_1^3+192\kappa_3\right),\ \alpha_2\to -\beta_1+\delta_{10}+\delta_{20},$$
$$\delta_2\to 0,\ \delta_{11}\to \frac{1}{192}\left(-1+96\beta_1-48\delta_{10}-96\delta_{20}+18\kappa_1-576\delta_{10}\kappa_1-192\kappa_1^3+192\kappa_3\right),$$
$$\omega_{11}\to \frac{1}{32}\left(-1+32\delta_{10}\right),\ \omega_{20}\to 0,\ \omega_2\to 0,\ \gamma_1\to \frac{1}{24}\left(1+24\beta_1\right),\ \gamma_{10}\to\beta_1,$$
$$\beta_{10}\to \frac{1}{24}\left(1+24\beta_1\right),\ \kappa_2\to \frac{1}{32}\left(-1+32\delta_{10}+32\kappa_1^2\right),\ \alpha_{10}\to 0,\ \alpha_1\to\delta_{10},$$
$$\left.\left.\delta_1\to 0,\ \omega_1\to 0,\ \omega_{10}\to 0,\ \gamma_0\to \frac{1}{4},\ \beta_0\to -\frac{1}{4},\ \kappa_0\to 1,\ \alpha_0\to 0,\ \delta_0\to 0,\ \omega_0\to 1\right\}\right\}$$

```
{V0, C0} /. sol[[1]]
```

$$\left\{\begin{pmatrix} 1+\frac{1}{32}c_1c_2\left(-1+32\delta_{10}\right)\hbar^2+\left(\frac{1}{64}c_1^2c_2\left(3\kappa_1-96\delta_{10}\kappa_1-32\kappa_1^3+32\kappa_3\right)+\frac{1}{64}c_1c_2^2\left(3\kappa_1-96\delta_{10}\kappa_1-\right.\right. \\ t[1] \\ t[2] \end{pmatrix}\right.$$

```
(V0 /. sol[[1]]) // Rot120 // dη[1] // dP[2 → 1]
```

$$\begin{pmatrix} 1+\frac{c_1\hbar}{4}+\frac{5}{96}c_1^2\hbar^2+\left(-\frac{c_1^3}{384}+\frac{1}{2}c_1^3\beta_1+c_1^3\beta_{11}-c_1^3\gamma_{20}+\frac{1}{2}c_1^3\delta_{10}+c_1^3\delta_{20}-\frac{3}{32}c_1^3\kappa_1+3c_1^3\delta_{10}\kappa_1+c_1^3\kappa_1^3-c\right. \\ t[1] \end{pmatrix}$$

```
(V0 /. sol[[1]]) // Rot120 // dη[1] // dP[2 → 1] // dA[1] // dcap[1]
```

$$\begin{pmatrix} 1+\frac{c_1\hbar}{4}+\frac{5}{96}c_1^2\hbar^2+\left(-\frac{c_1^3}{384}+\frac{1}{2}c_1^3\beta_1+c_1^3\beta_{11}-c_1^3\gamma_{20}+\frac{1}{2}c_1^3\delta_{10}+c_1^3\delta_{20}-\frac{3}{32}c_1^3\kappa_1+3c_1^3\delta_{10}\kappa_1+c_1^3\kappa_1^3-c\right. \\ t[1] \end{pmatrix}$$

```
indvars = Flatten[Union[Cases[Last /@ #, ϵ_k_ :> ϵ_k, Infinity]] & /@ sol]
```

$\{\beta_1, \beta_{11}, \gamma_3, \gamma_{12}, \gamma_{20}, \gamma_{21}, \gamma_{30}, \delta_{10}, \delta_{12}, \delta_{20}, \delta_{21}, \delta_{30}, \kappa_1, \kappa_3\}$

```
sol1 = Union[
  sol[[1]] /. Thread[indvars → 0],
  Thread[indvars → 0]
 ]
```

$$\left\{\alpha_0 \to 0,\ \alpha_1 \to 0,\ \alpha_2 \to 0,\ \alpha_3 \to -\frac{1}{64},\ \alpha_{10} \to 0,\ \alpha_{11} \to \frac{1}{192},\ \alpha_{12} \to 0,\ \alpha_{20} \to 0,\ \alpha_{21} \to -\frac{1}{192},\right.$$
$$\alpha_{30} \to 0,\ \beta_0 \to -\frac{1}{4},\ \beta_1 \to 0,\ \beta_2 \to \frac{1}{48},\ \beta_3 \to -\frac{1}{64},\ \beta_{10} \to \frac{1}{24},\ \beta_{11} \to 0,\ \beta_{12} \to -\frac{1}{192},$$
$$\beta_{20} \to 0,\ \beta_{21} \to -\frac{1}{192},\ \beta_{30} \to -\frac{1}{64},\ \gamma_0 \to \frac{1}{4},\ \gamma_1 \to \frac{1}{24},\ \gamma_2 \to \frac{1}{48},\ \gamma_3 \to 0,\ \gamma_{10} \to 0,$$
$$\gamma_{11} \to 0,\ \gamma_{12} \to 0,\ \gamma_{20} \to 0,\ \gamma_{21} \to 0,\ \gamma_{30} \to 0,\ \delta_0 \to 0,\ \delta_1 \to 0,\ \delta_2 \to 0,\ \delta_3 \to 0,\ \delta_{10} \to 0,$$
$$\delta_{11} \to -\frac{1}{192},\ \delta_{12} \to 0,\ \delta_{20} \to 0,\ \delta_{21} \to 0,\ \delta_{30} \to 0,\ \kappa_0 \to 1,\ \kappa_1 \to 0,\ \kappa_2 \to -\frac{1}{32},\ \kappa_3 \to 0,$$
$$\left.\omega_0 \to 1,\ \omega_1 \to 0,\ \omega_2 \to 0,\ \omega_3 \to 0,\ \omega_{10} \to 0,\ \omega_{11} \to -\frac{1}{32},\ \omega_{12} \to 0,\ \omega_{20} \to 0,\ \omega_{21} \to 0,\ \omega_{30} \to 0\right\}$$

```
V1 = V0 /. sol1
```

$$\begin{pmatrix} 1 - \frac{1}{32}(c_1 c_2)\hbar^2 + O[\hbar]^4 & h[1] \\ t[1] & \frac{1}{192} c_1 c_2 \hbar^2 + \left(-\frac{1}{384} c_1^2 c_2 - \frac{c_2^3}{384}\right)\hbar^3 + O[\hbar]^4 \quad -\frac{1}{4} + \frac{c_1 \hbar}{24} + \frac{1}{96} c_2^2 \hbar^2 + \left(-\frac{c_1^3}{384} - \frac{}{3}\right. \\ t[2] & \frac{1}{4} + \frac{c_2 \hbar}{24} + \frac{1}{96} c_2^2 \hbar^2 + O[\hbar]^4 \quad -\frac{1}{192}(c_1 \end{pmatrix}$$

```
C1 = C0 /. sol1
```

$$\begin{pmatrix} 1 - \frac{1}{64} c_1^2 \hbar^2 + O[\hbar]^4 \\ t[1] \end{pmatrix}$$

```
HardR4[V1]
```

True

```
TwistEq[V1]
```

True

```
V1 ** (V1 // dA[1] // dA[2])
```

(1)

```
CapEquation[V1, C1]
```

True

```
Φ1 = Φ[V1]
```

$$\begin{pmatrix} 1 & h[1] \\ t[1] & \left(\frac{1}{768}c_1c_2c_3-\frac{1}{576}c_2c_3^2\right)\hbar^3+O[\hbar]^4 \\ t[2] & -\frac{c_3\hbar}{48}+\left(\frac{c_2c_3}{96}+\frac{c_3^2}{192}\right)\hbar^2+\left(\frac{1}{768}c_1^2c_3-\frac{7c_2^2c_3}{2304}-\frac{c_1c_3^2}{2304}-\frac{5c_2c_3^2}{1152}-\frac{c_3^3}{2304}\right)\hbar^3+O[\hbar]^4 \\ t[3] & \frac{c_2\hbar}{24}+\left(-\frac{1}{96}c_1c_2+\frac{c_2c_3}{96}\right)\hbar^2+\left(\frac{1}{384}c_1^2c_2-\frac{c_2^3}{576}-\frac{7c_1c_2c_3}{2304}-\frac{1}{192}c_2^2c_3-\frac{5c_2c_3^2}{1152}\right)\hbar^3+O[\hbar]^4 \end{pmatrix}$$

$$\begin{matrix} & \frac{c_3\hbar}{16}+ \\ & \\ & \\ & -\frac{c_1\hbar}{16}+\left(\end{matrix}$$

```
Pentagon[Φ1]
```

True

```
Hexagon[+1, Φ1]
```

True

```
Hexagon[-1, Φ1]
```

True

```
Φ1 ** (Φ1 // dP[3, 2, 1])
```

(1)

```
Φ1 ** (Φ1 // dS[1] // dS[2] // dS[3])
```

$$\begin{pmatrix} 1 & h[1] & h[2] & h[3] \\ t[1] & 0 & \left(\frac{c_2c_3}{48}+\frac{c_3^2}{96}\right)\hbar^2+O[\hbar]^4 & \left(-\frac{1}{48}c_1c_2+\frac{c_2c_3}{48}\right)\hbar^2+O[\hbar]^4 \\ t[2] & \left(\frac{c_2c_3}{48}+\frac{c_3^2}{96}\right)\hbar^2+O[\hbar]^4 & 0 & \left(-\frac{c_1^2}{96}-\frac{c_1c_2}{48}\right)\hbar^2+O[\hbar]^4 \\ t[3] & \left(-\frac{1}{48}c_1c_2+\frac{c_2c_3}{48}\right)\hbar^2+O[\hbar]^4 & \left(-\frac{c_1^2}{96}-\frac{c_1c_2}{48}\right)\hbar^2+O[\hbar]^4 & 0 \end{pmatrix}$$