Pensieve header: β-calculations.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\2012-03"];
<< betaCalculus.m
```

The Knot-Theoretic Equations

R2, OC, R3 and easy R4

```
{R[1, 2] Ri[3, 4],
 R[1, 2] Ri[3, 4] // dm[1, 3, 1] // dm[2, 4, 2],
 R[1, 2] Ri[3, 4] // dm[1, 3, 1] // dm[4, 2, 2]
}
```

$$\left\{\begin{pmatrix} 1 & h[2] & h[4] \\ t[1] & \frac{-1+e^{c_1}}{c_1} & 0 \\ t[3] & 0 & \frac{e^{-c_3}(1-e^{c_3})}{c_3} \end{pmatrix}, (1), (1)\right\}$$

```
{
 R[1, 2] ** Ri[1, 2],
 R[1, 3] ** R[2, 3],
 R[1, 3] ** R[2, 3] == R[2, 3] ** R[1, 3] // Simplify,
 R[3, 1] ** R[3, 2] == R[3, 2] ** R[3, 1],
 R[1, 2] ** R[1, 3] ** R[2, 3],
 R[1, 2] ** R[1, 3] ** R[2, 3] == R[2, 3] ** R[1, 3] ** R[1, 2]
}
```

$$\left\{(1), \begin{pmatrix} 1 & h[3] \\ t[1] & \frac{-1+e^{c_1}}{c_1} \\ t[2] & \frac{-e^{c_1}+e^{c_1+c_2}}{c_2} \end{pmatrix}, \frac{(-1+e^{c_1})(-1+e^{c_2})}{c_1} == 0 \;\&\&\; \frac{(-1+e^{c_1})(-1+e^{c_2})}{c_2} == 0,\right.$$

$$\left.\text{True}, \begin{pmatrix} 1 & h[2] & h[3] \\ t[1] & \frac{-1+e^{c_1}}{c_1} & \frac{-1+e^{c_1}}{c_1} \\ t[2] & 0 & \frac{-e^{c_1}+e^{c_1+c_2}}{c_2} \end{pmatrix}, \text{True}\right\}$$

```
{
 R[3, 1] ** R[3, 2],
 R[3, 1],
 R[3, 1] // dΔ[1, 1, 2],
 R[3, 1] ** R[3, 2] == (R[3, 1] // dΔ[1, 1, 2])
}
```

$$\left\{\begin{pmatrix} 1 & h[1] & h[2] \\ t[3] & \frac{-1+e^{c_3}}{c_3} & \frac{-1+e^{c_3}}{c_3} \end{pmatrix}, \begin{pmatrix} 1 & h[1] \\ t[3] & \frac{-1+e^{c_3}}{c_3} \end{pmatrix}, \begin{pmatrix} 1 & h[1] & h[2] \\ t[3] & \frac{-1+e^{c_3}}{c_3} & \frac{-1+e^{c_3}}{c_3} \end{pmatrix}, \text{True}\right\}$$

```
R[1, 2, p1] ** R[1, 2, p2] == R[1, 2, p1 + p2] // Simplify
```

True

Hard R4

```
{
 R[1, 3] ** R[2, 3],
 R[1, 3] // dΔ[1, 1, 2],
 R[1, 3] ** R[2, 3] == (R[1, 3] // dΔ[1, 1, 2])
}
```

$$\left\{\begin{pmatrix} 1 & h[3] \\ t[1] & \frac{-1+e^{c_1}}{c_1} \\ t[2] & \frac{-e^{c_1}+e^{c_1+c_2}}{c_2} \end{pmatrix}, \begin{pmatrix} 1 & h[3] \\ t[1] & \frac{-1+e^{c_1+c_2}}{c_1+c_2} \\ t[2] & \frac{-1+e^{c_1+c_2}}{c_1+c_2} \end{pmatrix}, \frac{-1+e^{c_1}}{c_1} == \frac{-1+e^{c_1+c_2}}{c_1+c_2} \&\& \frac{-e^{c_1}+e^{c_1+c_2}}{c_2} == \frac{-1+e^{c_1+c_2}}{c_1+c_2}\right\}$$

```
{
 V0 = B[V0[c1, c2], Sum[V10 i+j[c1, c2] t[i] h[j], {i, 2}, {j, 2}]],
 R[1, 3] ** R[2, 3] ** V0,
 V0 ** (R[1, 3] // dΔ[1, 1, 2]),
 eqns1 = HardR4[V0]
}
```

$$\left\{\begin{pmatrix} V_0[c_1, c_2] & h[1] & h[2] \\ t[1] & V_{11}[c_1, c_2] & V_{12}[c_1, c_2] \\ t[2] & V_{21}[c_1, c_2] & V_{22}[c_1, c_2] \end{pmatrix}, \begin{pmatrix} V_0[c_1, c_2] & h[1] & h[2] & \\ t[1] & V_{11}[c_1, c_2] & V_{12}[c_1, c_2] & \frac{-1+e^{c_1}+e^{c_1} c_1 V_{12}[c_1, c_2]-}{} \\ t[2] & V_{21}[c_1, c_2] & V_{22}[c_1, c_2] & \frac{-e^{c_1}+e^{c_1+c_2}-e^{c_1} c_1 V_{12}[c_1, c_2}{} \end{pmatrix}\right.$$

$$\frac{1-e^{c_1+c_2}}{c_1+c_2} + \frac{(-1+e^{c_1})(1+c_2 V_{21}[c_1, c_2])}{c_1} == e^{c_1}(-1+e^{c_2}) V_{12}[c_1, c_2] \&\&$$
$$\left.\frac{1-e^{c_1+c_2}}{c_1+c_2} + \frac{e^{c_1}(-1+e^{c_2})(1+c_1 V_{12}[c_1, c_2])}{c_2} == (-1+e^{c_1}) V_{21}[c_1, c_2]\right\}$$

```
sol = Solve[eqns1 && V21[c1, c2] == 0, V12[c1, c2]]
```

$$\left\{\left\{V_{12}[c_1, c_2] \to -\frac{e^{-c_1}(-e^{c_1} c_1 + e^{c_1+c_2} c_1 + c_2 - e^{c_1} c_2)}{(-1+e^{c_2}) c_1 (c_1+c_2)}\right\}\right\}$$

```
V1 = V0 /. {V21[c1, c2] → 0, V11[c1, c2] → 0, V22[c1, c2] → 0, V0[c1, c2] → 1} /. sol[[1]]
```

$$\begin{pmatrix} 1 & h[2] \\ t[1] & \frac{e^{c_1} c_1 - e^{c_1+c_2} c_1 - c_2 + e^{c_1} c_2}{-e^{c_1} c_1^2 + e^{c_1+c_2} c_1^2 - e^{c_1} c_1 c_2 + e^{c_1+c_2} c_1 c_2} \\ t[2] & 0 \end{pmatrix}$$

Φ and the Pentagon

```
Φ1 = Φ[V1]
```

$$\begin{pmatrix} 1 & h[2] \\ t[1] & \frac{-e^{c_2} c_1 c_2 + e^{c_1+c_2} c_1 c_2 + e^{c_2+c_3} c_1 c_2 - e^{c_1+c_2+c_3} c_1 c_2 - e^{c_2} c_2^2 + e^{c_1+c_2} c_2^2 + e^{c_2+c_3} c_2^2 - e^{c_1+c_2+c_3} c_2^2 + c_1 c_3 - e^{c_2} c_1 c_3 - e^{c_1+c_2+c_3} c_1 c_3 + e^{c_1+2 c_2+c_3} c_1 c_3 - \cdots}{c_1^2 c_2 - e^{c_1+c_2} c_1^2 c_2 - e^{c_2+c_3} c_1^2 c_2 + e^{c_1+2 c_2+c_3} c_1^2 c_2 + c_1 c_2^2 - e^{c_1+c_2} c_1 c_2^2 - e^{c_2+c_3} c_1 c_2^2 + e^{c_1+2 c_2+c_3} c_1 c_2^2 + c_1 c_2 c_3 - e^{c_1+c_2} c_1 c_2 c_3 - \cdots} \\ t[2] & 0 \\ t[3] & 0 \end{pmatrix}$$

```
Pentagon[Φ1] // Simplify
```

True

```
V2 = V0 /. {V21 → 0, V11 → 0, V22 → 0, V0 → 1}
```

$$\begin{pmatrix} 1 & h[2] \\ t[1] & V_{12} \end{pmatrix}$$

```
Φ2 = Φ[V2]
```

$$\begin{pmatrix} 1 & h[3] \\ t[1] & 0 \\ t[2] & \frac{c_1 V_{12}^2}{1+c_1 V_{12}+c_2 V_{12}} \end{pmatrix}$$

```
Pentagon[Φ[V2]] // Simplify
```

$$\frac{c_1 c_3 V_{12}}{(1 + c_1 V_{12} + c_2 V_{12})\ (1 + c_1 V_{12} + c_2 V_{12} + c_3 V_{12})} == 0\ \&\&$$
$$\frac{c_1 c_2 V_{12}}{(1 + c_1 V_{12} + c_2 V_{12})\ (1 + c_1 V_{12} + c_2 V_{12} + c_3 V_{12})} == 0$$

Θ and the Hexagons

```
{Θ[1, 2],
 Θ[2, 1] == Θ[1, 2] // Simplify,
 (R[1, 2] // dP[1, 23]) ** R[2, 3] == R[2, 3] ** (R[1, 2] // dP[1, 23]),
 (R[2, 1] // dP[1, 23]) ** R[2, 3] == R[2, 3] ** (R[2, 1] // dP[1, 23]) // Simplify,
 (R[2, 1] // dP[1, 23]) ** Θ[2, 3] == Θ[2, 3] ** (R[2, 1] // dP[1, 23])  // Simplify
}
```

$$\left\{\begin{pmatrix} 1 & h[1] & h[2] \\ t[1] & \frac{-c_1+e^{\frac{c_2}{2}} c_1-e^{\frac{c_1}{2}+\frac{c_2}{2}} c_2+e^{\frac{c_2}{2}} c_2}{c_1^2+c_1 c_2} & \frac{-1+e^{\frac{c_1}{2}+\frac{c_2}{2}}}{c_1+c_2} \\ t[2] & \frac{-1+e^{\frac{c_1}{2}+\frac{c_2}{2}}}{c_1+c_2} & \frac{e^{\frac{c_1}{2}} c_1-e^{\frac{c_1}{2}+\frac{c_2}{2}} c_1-c_2+e^{\frac{c_1}{2}} c_2}{c_1 c_2+c_2^2} \end{pmatrix}, \text{True},\right.$$

$$\left.\text{True},\ \frac{(-1 + e^{c_2})\ (-1 + e^{c_2+c_3})\ c_3}{c_2\ (c_2 + c_3)} == 0\ \&\&\ \frac{(-1 + e^{c_2})\ (-1 + e^{c_2+c_3})}{c_2 + c_3} == 0,\ \text{True}\right\}$$

```
t1 = Hexagon[+1, Φ[V1]]
```

A very large output was generated. Here is a sample of it:

<<1>>

Show Less | Show More | Show Full Output | Set Size Limit...

```
Simplify[t1]
```

A very large output was generated. Here is a sample of it:

<<1>>

Show Less | Show More | Show Full Output | Set Size Limit...

The Twist Equation

```
V0
```

$$\begin{pmatrix} V_0[c_1, c_2] & h[1] & h[2] \\ t[1] & V_{11}[c_1, c_2] & V_{12}[c_1, c_2] \\ t[2] & V_{21}[c_1, c_2] & V_{22}[c_1, c_2] \end{pmatrix}$$

```
eqns2 = Simplify[(V0 // dP[2, 1]) ** Θ[1, 2] == R[1, 2] ** V0]
```

$$V_0[c_1, c_2] == V_0[c_2, c_1] \;\&\&\; \frac{1}{c_1\,(c_1 + c_2)}\left(-e^{\frac{c_2}{2}}\left(-1 + e^{\frac{c_1}{2}}\right) c_2\,(1 + c_2\,V_{12}[c_2, c_1]) + c_1\left(-1 + e^{\frac{c_2}{2}} - e^{\frac{c_2}{2}} c_2\left(\left(-1 + e^{\frac{c_1}{2}}\right) V_{12}[c_2, c_1] - V_{22}[c_2, c_1]\right)\right) + e^{\frac{c_2}{2}} c_1^2\,V_{22}[c_2, c_1]\right) ==$$

$$V_{11}[c_1, c_2] \;\&\&\; \frac{1}{c_1 + c_2}\left(-1 + e^{\frac{1}{2}(c_1+c_2)} + e^{\frac{1}{2}(c_1+c_2)} c_1\,V_{21}[c_2, c_1] + e^{\frac{1}{2}(c_1+c_2)} c_2\,V_{21}[c_2, c_1]\right) ==$$

$$\frac{1}{c_1}\left(e^{c_1} c_1\,V_{12}[c_1, c_2] + (-1 + e^{c_1})\,(1 + c_2\,V_{21}[c_1, c_2])\right) \;\&\&$$

$$\frac{1}{c_1 + c_2}\left(-1 + e^{\frac{1}{2}(c_1+c_2)} + e^{\frac{1}{2}(c_1+c_2)} c_1\,V_{12}[c_2, c_1] + e^{\frac{1}{2}(c_1+c_2)} c_2\,V_{12}[c_2, c_1]\right) == V_{21}[c_1, c_2] \;\&\&$$

$$(-1 + e^{c_1})\,V_{21}[c_1, c_2] + \frac{1}{c_2\,(c_1 + c_2)}\left(c_2\left(-1 + e^{\frac{c_1}{2}} + e^{\frac{c_1}{2}} c_2\,V_{11}[c_2, c_1]\right) - e^{\frac{c_1}{2}}\left(-1 + e^{\frac{c_2}{2}}\right) c_1^2\,V_{21}[c_2, c_1] - e^{\frac{c_1}{2}} c_1\left(-1 + e^{\frac{c_2}{2}} - c_2\left(V_{11}[c_2, c_1] - \left(-1 + e^{\frac{c_2}{2}}\right) V_{21}[c_2, c_1]\right)\right)\right) == e^{c_1}\,V_{22}[c_1, c_2]$$

The Non-Degeneracy Equations

```
eqns3 = Simplify[dη[1][V0] == B[1, 0] && dη[2][V0] == B[1, 0]]
```

$$V_0[0, c_2] == 1 \;\&\&\; V_{22}[0, c_2] == 0 \;\&\&\; V_0[c_1, 0] == 1 \;\&\&\; V_{11}[c_1, 0] == 0$$

Solving 1-3

```
eqns13 = eqns1 && eqns2 && eqns3
```

$$\frac{1 - e^{c_1+c_2}}{c_1 + c_2} + \frac{(-1 + e^{c_1})\,(1 + c_2\,V_{21}[c_1, c_2])}{c_1} == e^{c_1}\,(-1 + e^{c_2})\,V_{12}[c_1, c_2] \;\&\&$$

$$\frac{1 - e^{c_1+c_2}}{c_1 + c_2} + \frac{e^{c_1}\,(-1 + e^{c_2})\,(1 + c_1\,V_{12}[c_1, c_2])}{c_2} == (-1 + e^{c_1})\,V_{21}[c_1, c_2] \;\&\&$$

$$V_0[c_1, c_2] == V_0[c_2, c_1] \;\&\&\; \frac{1}{c_1\,(c_1 + c_2)}\left(-e^{\frac{c_2}{2}}\left(-1 + e^{\frac{c_1}{2}}\right) c_2\,(1 + c_2\,V_{12}[c_2, c_1]) + c_1\left(-1 + e^{\frac{c_2}{2}} - e^{\frac{c_2}{2}} c_2\left(\left(-1 + e^{\frac{c_1}{2}}\right) V_{12}[c_2, c_1] - V_{22}[c_2, c_1]\right)\right) + e^{\frac{c_2}{2}} c_1^2\,V_{22}[c_2, c_1]\right) ==$$

$$V_{11}[c_1, c_2] \;\&\&\; \frac{1}{c_1 + c_2}\left(-1 + e^{\frac{1}{2}(c_1+c_2)} + e^{\frac{1}{2}(c_1+c_2)} c_1\,V_{21}[c_2, c_1] + e^{\frac{1}{2}(c_1+c_2)} c_2\,V_{21}[c_2, c_1]\right) ==$$

$$\frac{1}{c_1}\left(e^{c_1} c_1\,V_{12}[c_1, c_2] + (-1 + e^{c_1})\,(1 + c_2\,V_{21}[c_1, c_2])\right) \;\&\&$$

$$\frac{1}{c_1 + c_2}\left(-1 + e^{\frac{1}{2}(c_1+c_2)} + e^{\frac{1}{2}(c_1+c_2)} c_1\,V_{12}[c_2, c_1] + e^{\frac{1}{2}(c_1+c_2)} c_2\,V_{12}[c_2, c_1]\right) == V_{21}[c_1, c_2] \;\&\&$$

$$(-1 + e^{c_1})\,V_{21}[c_1, c_2] + \frac{1}{c_2\,(c_1 + c_2)}\left(c_2\left(-1 + e^{\frac{c_1}{2}} + e^{\frac{c_1}{2}} c_2\,V_{11}[c_2, c_1]\right) - e^{\frac{c_1}{2}}\left(-1 + e^{\frac{c_2}{2}}\right) c_1^2\,V_{21}[c_2, c_1] - e^{\frac{c_1}{2}} c_1\left(-1 + e^{\frac{c_2}{2}} - c_2\left(V_{11}[c_2, c_1] - \left(-1 + e^{\frac{c_2}{2}}\right) V_{21}[c_2, c_1]\right)\right)\right) ==$$

$$e^{c_1}\,V_{22}[c_1, c_2] \;\&\&\; V_0[0, c_2] == 1 \;\&\&\; V_{22}[0, c_2] == 0 \;\&\&\; V_0[c_1, 0] == 1 \;\&\&\; V_{11}[c_1, 0] == 0$$

$$\text{Print /@ Solve}\Big[\frac{1-e^{c_1+c_2}}{c_1+c_2}+\frac{(-1+e^{c_1})\,(1+c_2\,V_{21}[c_1,\,c_2])}{c_1} == e^{c_1}\,(-1+e^{c_2})\,V_{12}[c_1,\,c_2]\;\&\&$$

$$\frac{1-e^{c_1+c_2}}{c_1+c_2}+\frac{e^{c_1}\,(-1+e^{c_2})\,(1+c_1\,V_{12}[c_1,\,c_2])}{c_2} == (-1+e^{c_1})\,V_{21}[c_1,\,c_2]\;\&\&$$

$$\frac{1}{c_1\,(c_1+c_2)}\Big(-e^{\frac{c_2}{2}}\Big(-1+e^{\frac{c_1}{2}}\Big)\,c_2\,(1+c_2\,V_{12}[c_2,\,c_1])+$$

$$c_1\Big(-1+e^{\frac{c_2}{2}}-e^{\frac{c_2}{2}}\,c_2\,\Big(\Big(-1+e^{\frac{c_1}{2}}\Big)\,V_{12}[c_2,\,c_1]-V_{22}[c_2,\,c_1]\Big)\Big)+e^{\frac{c_2}{2}}\,c_1^2\,V_{22}[c_2,\,c_1]\Big) ==$$

$$V_{11}[c_1,\,c_2]\;\&\&\;\frac{1}{c_1+c_2}\Big(-1+e^{\frac{1}{2}(c_1+c_2)}+e^{\frac{1}{2}(c_1+c_2)}\,c_1\,V_{21}[c_2,\,c_1]+e^{\frac{1}{2}(c_1+c_2)}\,c_2\,V_{21}[c_2,\,c_1]\Big) ==$$

$$\frac{1}{c_1}\,(e^{c_1}\,c_1\,V_{12}[c_1,\,c_2]+(-1+e^{c_1})\,(1+c_2\,V_{21}[c_1,\,c_2]))\;\&\&$$

$$\frac{1}{c_1+c_2}\Big(-1+e^{\frac{1}{2}(c_1+c_2)}+e^{\frac{1}{2}(c_1+c_2)}\,c_1\,V_{12}[c_2,\,c_1]+e^{\frac{1}{2}(c_1+c_2)}\,c_2\,V_{12}[c_2,\,c_1]\Big) == V_{21}[c_1,\,c_2]\;\&\&$$

$$(-1+e^{c_1})\,V_{21}[c_1,\,c_2]+\frac{1}{c_2\,(c_1+c_2)}\Big(c_2\Big(-1+e^{\frac{c_1}{2}}+e^{\frac{c_1}{2}}\,c_2\,V_{11}[c_2,\,c_1]\Big)-e^{\frac{c_1}{2}}\Big(-1+e^{\frac{c_2}{2}}\Big)\,c_1^2$$

$$V_{21}[c_2,\,c_1]-e^{\frac{c_1}{2}}\,c_1\Big(-1+e^{\frac{c_2}{2}}-c_2\Big(V_{11}[c_2,\,c_1]-\Big(-1+e^{\frac{c_2}{2}}\Big)\,V_{21}[c_2,\,c_1]\Big)\Big)\Big) ==$$

$$e^{c_1}\,V_{22}[c_1,\,c_2],\;\{V_{11}[c_2,\,c_1],\;V_{12}[c_2,\,c_1],\;V_{21}[c_2,\,c_1],\;V_{22}[c_2,\,c_1]\}\Big][[1]];$$

$$V_{11}[c_2,\,c_1]\to\frac{1}{c_2\,(c_1+c_2)}\Big(e^{\frac{c_1}{2}}\,c_1-e^{\frac{c_1}{2}+\frac{c_2}{2}}\,c_1-c_2+e^{\frac{c_1}{2}}\,c_2+e^{\frac{c_1}{2}}\,c_1^2\,V_{12}[c_1,\,c_2]-e^{\frac{c_1}{2}+\frac{c_2}{2}}\,c_1^2\,V_{12}[c_1,\,c_2]+$$

$$e^{\frac{c_1}{2}}\,c_1\,c_2\,V_{12}[c_1,\,c_2]-e^{\frac{c_1}{2}+\frac{c_2}{2}}\,c_1\,c_2\,V_{12}[c_1,\,c_2]+e^{\frac{c_1}{2}}\,c_1\,c_2\,V_{22}[c_1,\,c_2]+e^{\frac{c_1}{2}}\,c_2^2\,V_{22}[c_1,\,c_2]\Big)$$

$$V_{12}[c_2,\,c_1]\to-\frac{1}{c_1+c_2}e^{-\frac{c_1}{2}-\frac{c_2}{2}}\Big(-1+e^{\frac{c_1}{2}+\frac{c_2}{2}}-c_1\,V_{21}[c_1,\,c_2]-c_2\,V_{21}[c_1,\,c_2]\Big)$$

$$V_{21}[c_2,\,c_1]\to\frac{1}{c_1+c_2}\Big(-1+e^{\frac{c_1}{2}+\frac{c_2}{2}}+e^{\frac{c_1}{2}+\frac{c_2}{2}}\,c_1\,V_{12}[c_1,\,c_2]+e^{\frac{c_1}{2}+\frac{c_2}{2}}\,c_2\,V_{12}[c_1,\,c_2]\Big)$$

$$V_{22}[c_2,\,c_1]\to$$

$$-\frac{1}{c_1\,(c_1+c_2)}\,e^{-\frac{c_1}{2}-\frac{c_2}{2}}\Big(-e^{\frac{c_1}{2}}\,c_1+e^{\frac{c_1}{2}+\frac{c_2}{2}}\,c_1+c_2-e^{\frac{c_1}{2}}\,c_2-e^{\frac{c_1}{2}}\,c_1^2\,V_{11}[c_1,\,c_2]-e^{\frac{c_1}{2}}\,c_1\,c_2\,V_{11}[c_1,\,c_2]+$$

$$c_1\,c_2\,V_{21}[c_1,\,c_2]-e^{\frac{c_1}{2}}\,c_1\,c_2\,V_{21}[c_1,\,c_2]+c_2^2\,V_{21}[c_1,\,c_2]-e^{\frac{c_1}{2}}\,c_2^2\,V_{21}[c_1,\,c_2]\Big)$$

$$\text{eqns} = \frac{1-e^{c_1+c_2}}{c_1+c_2} + \frac{(-1+e^{c_1})\,(1+c_2\,V_{21}[c_1,\,c_2])}{c_1} == e^{c_1}\,(-1+e^{c_2})\,V_{12}[c_1,\,c_2]\;\&\&$$

$$\frac{1-e^{c_1+c_2}}{c_1+c_2} + \frac{e^{c_1}\,(-1+e^{c_2})\,(1+c_1\,V_{12}[c_1,\,c_2])}{c_2} == (-1+e^{c_1})\,V_{21}[c_1,\,c_2]\;\&\&$$

$$\frac{1}{c_1\,(c_1+c_2)}\left(-e^{\frac{c_2}{2}}\left(-1+e^{\frac{c_1}{2}}\right)c_2\,(1+c_2\,V_{12}[c_2,\,c_1]) + c_1\left(-1+e^{\frac{c_2}{2}}-e^{\frac{c_2}{2}}\,c_2\left(\left(-1+e^{\frac{c_1}{2}}\right)V_{12}[c_2,\,c_1]-V_{22}[c_2,\,c_1]\right)\right)+e^{\frac{c_2}{2}}\,c_1^2\,V_{22}[c_2,\,c_1]\right) ==$$

$$V_{11}[c_1,\,c_2]\;\&\&\;\frac{1}{c_1+c_2}\left(-1+e^{\frac{1}{2}(c_1+c_2)}+e^{\frac{1}{2}(c_1+c_2)}\,c_1\,V_{21}[c_2,\,c_1]+e^{\frac{1}{2}(c_1+c_2)}\,c_2\,V_{21}[c_2,\,c_1]\right) ==$$

$$\frac{1}{c_1}\,(e^{c_1}\,c_1\,V_{12}[c_1,\,c_2]+(-1+e^{c_1})\,(1+c_2\,V_{21}[c_1,\,c_2]))\;\&\&$$

$$\frac{1}{c_1+c_2}\left(-1+e^{\frac{1}{2}(c_1+c_2)}+e^{\frac{1}{2}(c_1+c_2)}\,c_1\,V_{12}[c_2,\,c_1]+e^{\frac{1}{2}(c_1+c_2)}\,c_2\,V_{12}[c_2,\,c_1]\right) == V_{21}[c_1,\,c_2]\;\&\&$$

$$(-1+e^{c_1})\,V_{21}[c_1,\,c_2]+\frac{1}{c_2\,(c_1+c_2)}\left(c_2\left(-1+e^{\frac{c_1}{2}}+e^{\frac{c_1}{2}}\,c_2\,V_{11}[c_2,\,c_1]\right)-e^{\frac{c_1}{2}}\left(-1+e^{\frac{c_2}{2}}\right)c_1^2\,V_{21}[c_2,\right.$$

$$\left.c_1]-e^{\frac{c_1}{2}}\,c_1\left(-1+e^{\frac{c_2}{2}}-c_2\left(V_{11}[c_2,\,c_1]-\left(-1+e^{\frac{c_2}{2}}\right)V_{21}[c_2,\,c_1]\right)\right)\right) == e^{c_1}\,V_{22}[c_1,\,c_2];$$

$$\text{Solve}[\text{eqns}\;\&\&\;(\text{eqns}\;/.\;\{c_1\to c_2,\;c_2\to c_1\})\;\&\&\;V_{12}[c_2,\,c_1] == V_{12}[c_1,\,c_2],$$
$$\{V_{11}[c_1,\,c_2],\;V_{12}[c_1,\,c_2],\;V_{21}[c_1,\,c_2],\;V_{22}[c_1,\,c_2],$$
$$V_{11}[c_2,\,c_1],\;V_{12}[c_2,\,c_1],\;V_{21}[c_2,\,c_1],\;V_{22}[c_2,\,c_1]\}][[1]]$$

Solve::svars : Equations may not give solutions for all "solve" variables. ≫

$$\left\{V_{12}[c_1,\,c_2]\to-\frac{1}{c_1+c_2},\;V_{21}[c_1,\,c_2]\to-\frac{1}{c_1+c_2},\;V_{11}[c_2,\,c_1]\to\frac{-1+e^{\frac{c_1}{2}}}{c_1+c_2}+e^{\frac{c_1}{2}}\,V_{22}[c_1,\,c_2],\right.$$

$$\left.V_{12}[c_2,\,c_1]\to-\frac{1}{c_1+c_2},\;V_{21}[c_2,\,c_1]\to-\frac{1}{c_1+c_2},\;V_{22}[c_2,\,c_1]\to-\frac{e^{-\frac{c_2}{2}}\left(-1+e^{\frac{c_2}{2}}\right)}{c_1+c_2}+e^{-\frac{c_2}{2}}\,V_{11}[c_1,\,c_2]\right\}$$

$$\text{Solve}\left[\text{eqns}\;\&\&\;(\text{eqns}\;/.\;\{c_1\to c_2,\;c_2\to c_1\})\;\&\&\;V_{12}[c_2,\,c_1] == V_{12}[c_1,\,c_2]\;\&\&\right.$$
$$V_{22}[c_1,\,c_2] == 0\;\&\&\left(V_{11}[c_1,\,c_2] == \frac{-1+e^{\frac{c_2}{2}}}{c_1+c_2}\right),\;\{V_{11}[c_1,\,c_2],\;V_{12}[c_1,\,c_2],$$
$$\left.V_{21}[c_1,\,c_2],\;V_{22}[c_1,\,c_2],\;V_{11}[c_2,\,c_1],\;V_{12}[c_2,\,c_1],\;V_{21}[c_2,\,c_1],\;V_{22}[c_2,\,c_1]\}\right][[1]]$$

$$\left\{V_{11}[c_1,\,c_2]\to\frac{-1+e^{\frac{c_2}{2}}}{c_1+c_2},\;V_{12}[c_1,\,c_2]\to-\frac{1}{c_1+c_2},\;V_{21}[c_1,\,c_2]\to-\frac{1}{c_1+c_2},\;V_{22}[c_1,\,c_2]\to0,\right.$$

$$\left.V_{11}[c_2,\,c_1]\to\frac{-1+e^{\frac{c_1}{2}}}{c_1+c_2},\;V_{12}[c_2,\,c_1]\to-\frac{1}{c_1+c_2},\;V_{21}[c_2,\,c_1]\to-\frac{1}{c_1+c_2},\;V_{22}[c_2,\,c_1]\to0\right\}$$

```
V2 = V0 /. {V11[c1, c2] → (-1 + e^(c2/2))/(c1 + c2), V12[c1, c2] → -1/(c1 + c2),
    V21[c1, c2] → -1/(c1 + c2), V22[c1, c2] → 0, V0[__] → 1}
```

$$\begin{pmatrix} 1 & h[1] & h[2] \\ t[1] & \frac{-1+e^{\frac{c_2}{2}}}{c_1+c_2} & \frac{1}{-c_1-c_2} \\ t[2] & \frac{1}{-c_1-c_2} & 0 \end{pmatrix}$$

```
Φ[V2]
```

Power::infy : Infinite expression $\frac{1}{0}$ encountered. ≫

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General::stop : Further output of Power::infy will be suppressed during this calculation. ≫

Infinity::indet :
  Indeterminate expression ComplexInfinity + ComplexInfinity + ComplexInfinity + ComplexInfinity encountered. ≫

Infinity::indet :
  Indeterminate expression ComplexInfinity + ComplexInfinity + ComplexInfinity + ComplexInfinity encountered. ≫

$$\begin{pmatrix} 1 \end{pmatrix}$$

```
HardR4[V2]
```

True

```
Simplify[{
   (R[1, 3] ** R[2, 3] ** V2), (V2 ** (R[1, 3] // dΔ[1, 1, 2]))
  }]
```

$$\left\{\begin{pmatrix} 1 & h[1] & h[2] & h[3] \\ t[1] & \frac{-1+e^{\frac{c_2}{2}}}{c_1+c_2} & \frac{1}{-c_1-c_2} & \frac{-1+e^{c_1+c_2}}{c_1+c_2} \\ t[2] & \frac{1}{-c_1-c_2} & 0 & \frac{-1+e^{c_1+c_2}}{c_1+c_2} \end{pmatrix}, \begin{pmatrix} 1 & h[1] & h[2] & h[3] \\ t[1] & \frac{-1+e^{\frac{c_2}{2}}}{c_1+c_2} & \frac{1}{-c_1-c_2} & \frac{-1+e^{c_1+c_2}}{c_1+c_2} \\ t[2] & \frac{1}{-c_1-c_2} & 0 & \frac{-1+e^{c_1+c_2}}{c_1+c_2} \end{pmatrix}\right\}$$

```
Simplify[(V2 // dP[2, 1]) ** Θ[1, 2] == R[1, 2] ** V2]
```

True

```
Simplify[{(V2 // dP[2, 1]) ** Θ[1, 2], R[1, 2] ** V2}]
```

$$\left\{\begin{pmatrix} 1 & h[1] & h[2] \\ t[1] & \frac{-1+e^{\frac{c_2}{2}}}{c_1+c_2} & \frac{1}{-c_1-c_2} \\ t[2] & \frac{1}{-c_1-c_2} & \frac{-1+e^{c_1}}{c_1+c_2} \end{pmatrix}, \begin{pmatrix} 1 & h[1] & h[2] \\ t[1] & \frac{-1+e^{\frac{c_2}{2}}}{c_1+c_2} & \frac{1}{-c_1-c_2} \\ t[2] & \frac{1}{-c_1-c_2} & \frac{-1+e^{c_1}}{c_1+c_2} \end{pmatrix}\right\}$$

```
dη[1][V2]
```

$$\begin{pmatrix} 1 \end{pmatrix}$$

```
dη[2][V2]
(1)
```