Projective Modules

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$$\begin{tikzcd} & B \arrow[d] \\ P \arrow[r] \arrow[ur, dashed, "\exists"] & A \end{tikzcd}$$

True if P is free

$$\begin{tikzcd} & P \arrow[r, shift left] \arrow[dl] \arrow[d] & F \arrow[l, shift left] \arrow[dll] \arrow[dl] & \\ B \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

True if

$P \rightleftarrows F$ (F free)

Converse?

<u>Claim</u> If $P \underset{\pi}{\overset{\iota}{\rightleftarrows}} F$ then

$$F = P \oplus (\ker \pi).$$

Q. What is the fundamental reason why this is significant?

<u>Claim</u> If $0 \to P \to Q \to R \to 0$ is exact, then $Q = P \oplus R$.

<u>Pf</u> By projectivity of R,

$$\begin{tikzcd} & R \arrow[dl, "\exists\sigma"'] \arrow[d, "I"] & \\ Q \arrow[r] & R \arrow[r] & 0 \end{tikzcd}$$

the sequence splits:

$$0 \longrightarrow P \longrightarrow Q \underset{}{\overset{\sigma}{\leftrightarrows}} R \longrightarrow 0$$

We've only used the projectivity of R here;

in fact, we've only used that every surjection on P has a section. Converse?

<u>Claim</u> If every surjection on P has a section, then P is projective.

$$\begin{tikzcd} & P \arrow[d] & & ? \\ B \arrow[r] & A \arrow[r] & 0 & \end{tikzcd}$$