Staudacher: Integrability and the AdS/CFT Correspondence

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Integrability: "#DoF = = # cons. quantities"

"Always related to YB".

What is AdS/CFT?

CFT: Conformal Field Theory in dim. 1+3.

Precisely, $N=4$ Supersymmetric Yang-Mills QFT.

AdS: stands for a $5+5=10$ dim product space $AdS_5 \times S^5$

AdS_5 → 5-d Anti-de-Sitter space

S^5 → the 5-sphere

IIB superstring embedded in this space.

Conjecture (Maldacena, 1997) "AdS" = "CFT".

Quantum Integrability (Bethe, 1931) — the one-dimensional Heisenberg spin chain.

Spin $\frac{1}{2}$ in QM: Electron: $|x, s\rangle$ $s \in \pm\frac{1}{2}$

"the spin part" of $\mathbb{H}$ is $\mathbb{C}^2$.

"The spin operator" $\vec{S} = \frac{1}{2}\vec{\sigma}$

$\sigma^1 = \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \sigma^2 = \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$\sigma^3 = \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$[\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k$$

so $[S^i, S^j] = i\epsilon^{ijk}S^k$

"the simplest non-trivial representation of $su(2)$".

$\{$unitary irreps of $su(2)\}$ $= \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots\}$

singlet rep

Ladder operators:

$S^\pm = S_1 \pm iS_2 \qquad [S^3, S^\pm] = \pm S^\pm, \ [S^+, S^-] = 2S^3$

$S^+ = \sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad S^- = \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$U = e^{i\vec{\alpha}\cdot\vec{S}}$ must be unitary: $UU^\dagger = 1$.

Act w/ $S^\pm$ & S^3 on electron:

$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ a basis in $\mathbb{C}^2$.

$S^3|\uparrow\rangle = \frac{1}{2}|\uparrow\rangle \qquad S^3|\downarrow\rangle = -\frac{1}{2}|\downarrow\rangle$

$\underbrace{S^+|\uparrow\rangle}_{} = 0 \qquad \underbrace{S^-}_{}|\uparrow\rangle = |\downarrow\rangle$

$$\underbrace{S^+|\uparrow\rangle}_{\substack{\text{highest}\\ \text{weight}\\ \text{state}}} = 0 \qquad \underbrace{S^-}_{\substack{\text{lowering}\\ \text{operator.}}}|\uparrow\rangle = |\downarrow\rangle$$

$$\underbrace{S^+}_{\substack{\text{"raising}\\ \text{op"}}}|\downarrow\rangle = |\uparrow\rangle \qquad S^-\underbrace{|\downarrow\rangle}_{\substack{\text{"lowest}\\ \text{state"}}} = 0$$

Make a 1D metal:

$\uparrow \downarrow \uparrow \uparrow \uparrow \ldots\ldots$ "spin chain" of length L

$$\mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2 \cong \mathbb{C}^{(2^L)}$$

$$\vec{S} = \sum_{l=1}^{L} \vec{S}_l \qquad \vec{S}_l = 1 \otimes \ldots \otimes S_l \otimes \ldots \otimes 1$$

$$\tfrac{1}{2} \otimes \tfrac{1}{2} = 1 \oplus 0 \iff \underline{2} \otimes \underline{2} = \underline{3} \oplus \underline{1}$$

In our basis, $1 = \{|\uparrow\uparrow\rangle, \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle), |\downarrow\downarrow\rangle\}$

$$0 = \{\tfrac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)\}$$

$$\underline{2} \otimes \underline{2} \otimes \ldots \otimes \underline{2} = \underline{L+1} \oplus \ldots$$

$$\tfrac{1}{2} \otimes \ldots \otimes \tfrac{1}{2} = \tfrac{L}{2} \oplus \ldots$$

The Heisenberg Hamiltonian

$$\hat{H} = 4 \sum_{l=1}^{L} \left(\tfrac{1}{4} - \vec{S}_l \cdot \vec{S}_{l+1}\right) \qquad \left(\substack{\text{setting}\\ \vec{S}_{L+1} = \vec{S}_1}\right)$$

a $2^L \times 2^L$ matrix.

We want the spectrum of this matrix, & to

solve $\hat{H}|\Psi\rangle = E|\Psi\rangle$.

Exercise 1 Write $\hat{H}$ for $L=2,3,4$ & diagonalize.

Exercise 2 $[\hat{H}, \vec{S}] = 0$.

total spin