Braverman: Index Theory and Geometric Quantization of Non-Compact Manifolds

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10:03 AM

Today: Introduction to Atiyah-Singer.

$A: V_1 \to V_2$ a linear trans between v.s.

$\dim \ker A = \#\{\lambda_i(A) = 0\}$ may jump up within a cont. family.

$\operatorname{Coker} A = V_2 / \operatorname{im} A$.

Claim

$$\dim \ker A - \dim \operatorname{coker} A = \dim V_1 - V_2$$

hence l.h.s. is indep of A.

Now $A: V_1 \to V_2$, $V_{1,2}$ Hilbert spaces.

Definition. A is "Fredholm" if

$$\dim A < \infty, \quad \dim \operatorname{coker} A < \infty.$$

In that case, set

$$\operatorname{ind} A = \dim \ker A - \dim \operatorname{coker} A.$$

to be defined.

Thm 1. (Fritz Noether) In a reasonable cont. family of Fredholm operators, the index is constant.

roughly, $K \sim \begin{pmatrix} \lambda_1 & \\ & \ddots \end{pmatrix}$ w/ $\lambda_i \to 0$.

2. If K is compact then $\operatorname{ind} A = \operatorname{ind}(A+K)$

Example

$$T: \mathcal{H} \to \mathcal{H}, \quad \mathcal{H} = \ell^2(\mathbb{N})$$

$T(x_1, \ldots) = (x_3, x_4, \ldots)$

$\dim\ker = 2T$ $\quad$ $\dim \operatorname{coker} T = 0$

$\operatorname{ind} T = 2$, could not happen in F.D.

Throw in a compact group G, acting on V_1, V_2, w/ an equivariant A.

$\Rightarrow \ker A$ is invariant under G, G acts on $\operatorname{coker} A$.

V_0: always the trivial rep.

$\ker A =$ something like $3V_1 \oplus 2V_2 + V_3$ $\quad$ V_i irreps.

$$= \sum_{m_i \in \mathbb{Z}_{\geq 0}} m_i^+ V_i$$

$$\operatorname{coker} A = \sum m_i^- V_i$$

Set

$$\operatorname{ind}_G A = (m_1^+ - m_1^-, \ldots\ldots)$$

or better,

$$\operatorname{ind}_G A = \ker A - \operatorname{coker} A \in R(G)$$
$$= \{V - U\} / \text{if } {V-U = A-B \atop V \oplus B = U \oplus A}$$
$$= \text{the Grothendieck Group of } \operatorname{Rep}(G)$$

Remark $\hat{R}(G) := \left\{ \sum m_i V_i : \begin{array}{l} m_i \in \mathbb{Z} \text{ finite,} \\ V_i \text{ irreps,} \\ \text{summation may} \\ \text{be infinite} \end{array} \right\}$

Example $G = \mathbb{Z}_2 \curvearrowright V = \ell^2$ by

$g : \ell^2 \to \ell^2$, $g(x_1, x_2, x_3, \ldots) = (x_2, x_1, x_4, x_3, \ldots)$

where g is the non-trivial element of $\mathbb{Z}_2$

w/ T as before,

$$\ker T = V_0 \oplus V_1 \quad \text{where on } V_1,\ g = -1.$$

so $\operatorname{ind} T = V_0 \oplus V_1$

Consider differential operators:

$$\mathcal{D} = \sum a_{ij} \frac{\partial^2}{\partial x^i \partial x^j} + \text{lower orders}$$

acting on Sobolev spaces

$$H_0^2(\mathbb{R}^n) = \{ f : \underbrace{f, \partial f, \partial\partial f \in L^2}_{\text{compact support}} \}$$

$\mathcal{D}$ is a bounded operator

$$\mathcal{D} : H_0^2 \longrightarrow L^2.$$

Likewise for operators on compact manifolds...

The symbol of $\mathcal{D}$ ...

<u>Def</u> $\mathcal{D}$ is elliptic if $\sigma_k(\xi) \neq 0$ for $\xi \neq 0$.

In the bundle case, $\mathcal{D}$ is elliptic if its symbol is invertible for $\xi \neq 0$.

<u>Theorem</u> 1. Eliptic regularity:

$$\mathcal{D} f = u \in C^\infty \Rightarrow f \in C^\infty.$$

2. $\mathcal{D}$ is Fredholm.

3. If ord $\mathcal{D} = k$, $\mathcal{D}: H^k \to L^2$, A is an operator of order $< k$, then

$$A : H^k \longrightarrow L^2 \text{ is compact.}$$

$\Longrightarrow$ The index of $\mathcal{D}$ depends just on its symbol.

Question by Gel'fand: There has to be a topological way to determine the index from the symbol?