Poisson Groups and Lie BiAlgebras

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From Drinfeld's quantum groups paper:

We shall need a Hopf algebra version of the above definition. In this case A_0 is a Poisson-Hopf algebra (i.e. a Hopf algebra structure and a Poisson algebra structure on A_0 are given such that the multiplication is the same for both structures and the comultiplication $A_0 \to A_0 \otimes A_0$ is a Poisson algebra homomorphism, the Poisson bracket on $A_0 \otimes A_0$ being defined by $\{a \otimes b, c \otimes d\} = ac \otimes \{b, d\} + \{a, c\} \otimes bd$) and A is a Hopf algebra deformation of A_0. We shall also use the dual notion of quantization of co-Poisson-Hopf algebras (a co-Poisson-Hopf algebra is a cocommutative Hopf algebra B with a Poisson cobracket $B \to B \otimes B$ compatible with the Hopf algebra structure).

We discuss the structure of Poisson-Hopf algebras and co-Poisson-Hopf algebras in §§3 and 4. Then we consider the quantization problem.

3. Poisson groups and Lie bialgebras. A *Poisson group* is a group G with a Poisson bracket on $\mathrm{Fun}(G)$ which makes $\mathrm{Fun}(G)$ a Poisson-Hopf algebra. In other words the Poisson bracket must be compatible with the group operation, which means that the mapping $\mu \colon G \times G \to G$, $\mu(g_1, g_2) = g_1 g_2$, must be a Poisson mapping in the sense of **[33]**, i.e. $\mu^* \colon \mathrm{Fun}(G) \to \mathrm{Fun}(G \times G)$ must be a Lie algebra homomorphism. Specifying the meaning of the word "group" and the symbol $\mathrm{Fun}(G)$, we obtain the notions of Poisson-Lie group, Poisson formal group, Poisson algebraic group, etc. According to our general principles the notions of Poisson group and Poisson-Hopf algebra are equivalent.

There exists a very simple description of Poisson-Lie groups in terms of *Lie bialgebras.*

DEFINITION. A *Lie bialgebra* is a vector space $\mathfrak{g}$ with a Lie algebra structure and a Lie coalgebra structure, these structures being compatible in the following sense: the cocommutator mapping $\mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ must be a 1-cocycle ($\mathfrak{g}$ acts on $\mathfrak{g} \otimes \mathfrak{g}$ by means of the adjoint representation).

If G is a Poisson-Lie group then $\mathfrak{g} = \mathrm{Lie}\,(G)$ has a Lie bialgebra structure. To define it write the Poisson bracket on $C^\infty(G)$ as

$$\{\varphi, \psi\} = \eta^{\mu\nu} \partial_\mu \varphi \cdot \partial_\nu \psi, \qquad \varphi, \psi \in C^\infty(G), \tag{4}$$

where $\{\partial_\mu\}$ is a basis of right-invariant vector fields on G. The compatibility of the bracket with the group operation means that the function $\eta \colon G \to \mathfrak{g} \otimes \mathfrak{g}$ corresponding to $\eta^{\mu\nu}$ is a 1-cocycle. The 1-cocycle $f \colon \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ corresponding to η defines a Lie bialgebra structure on $\mathfrak{g}$ (the Jacobi identity for $f^* \colon \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ holds because f^* is the infinitesimal part of the bracket (4)).

THEOREM 1. *The category of connected and simply-connected Poisson-Lie groups is equivalent to the category of finite dimensional Lie bialgebras.*

The analogue of Theorem 1 for Poisson formal groups over a field of characteristic 0 can be proved in the following way. The algebra of functions on the formal group corresponding to $\mathfrak{g}$ is nothing but $(U\mathfrak{g})^*$. A Poisson-Hopf structure on $(U\mathfrak{g})^*$ is equivalent to a co-Poisson-Hopf structure on $U\mathfrak{g}$. So it suffices to prove the following easy theorem.

THEOREM 2. *Let $\delta \colon U\mathfrak{g} \to U\mathfrak{g} \otimes U\mathfrak{g}$ be a Poisson cobracket which makes $U\mathfrak{g}$ a co-Poisson-Hopf algebra (the Hopf structure on $U\mathfrak{g}$ is usual). Then $\delta(\mathfrak{g}) \subset \mathfrak{g} \otimes \mathfrak{g}$ and $(\mathfrak{g}, \delta|\mathfrak{g})$ is a Lie bialgebra. Thus we obtain a one-to-one correspondence between co-Poisson-Hopf structures on $U\mathfrak{g}$ inducing the usual Hopf structure and Lie bialgebra structures on $\mathfrak{g}$ inducing the given Lie algebra structure.*

Question. Is there a diagrammatic version of that?