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Cosmic Coincidences and Several Other Stories, 1

Dror Bar-Natan at the University of Tennessee

March 4, 2011, http://www.math.toronto.edu/~drorbn/Talks/Tennessee-1103/

Abstract. In the first half of my talk I will tell a cute and simple story — how given a knot in $\mathbb{R}^3$ one may count all possible "cosmic coincidences" associated with that knot, and how this count, appropriately packaged, becomes an invariant Z with values in some space $\mathcal{A}$ of linear combinations of certain trivalent graphs.

In the second half of my talk I will describe (rather sketchily, I'm afraid) a part of the story surrounding Z and $\mathcal{A}$: How the same Z also comes from quantum field theory, Feynman diagrams, and configuration space integrals. How $\mathcal{A}$ is a space of universal formulas which make sense in every metrized Lie algebra and how specific choices for that Lie algebra correspond to various famed knot invariants. How Z solves a universal topological problem, and how solving for Z is solving some universal Lie-algebraic problem. All together, this is the u-story.

In the remaining time I will mention several other Z's and $\mathcal{A}$'s and the parallel (yet sometimes interwoven) stories surrounding them — the v-story, and w-story, and perhaps also the p-story. Each of these stories is clearly still missing some chapters.

Creation of Adam

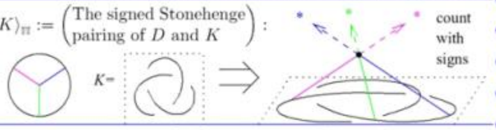
Michelangelo

improve

Disclaimer

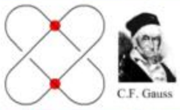
We'll concentrate on the beauty and ignore the cracks.

$\langle D, K\rangle_{\text{ππ}} :=$ (The signed Stonehenge pairing of D and K) : count with signs



The Gaussian linking number

$lk(\text{◯◯}) = \sum_{\text{vertical chopsticks}} (\text{signs}) = \langle \text{O–O}, \text{◯◯} \rangle_{\text{ππ}}$



The generating function of all cosmic coincidences:

$$Z(K) := \lim_{N\to\infty} \sum_{\text{3-valent } D} \frac{\langle D, K\rangle_{\text{ππ}} D}{2^c c! \binom{N}{e}} \cdot \begin{pmatrix}\text{framing-}\\ \text{dependent}\\ \text{counter-term}\end{pmatrix} \in \mathcal{A}(\circlearrowleft)$$

N := # of stars

c := # of chopsticks

e := # of edges of D

$\mathcal{A}(\circlearrowleft)$:=Span $\langle$  $\rangle$ / oriented vertices, AS:  =0 & more relations

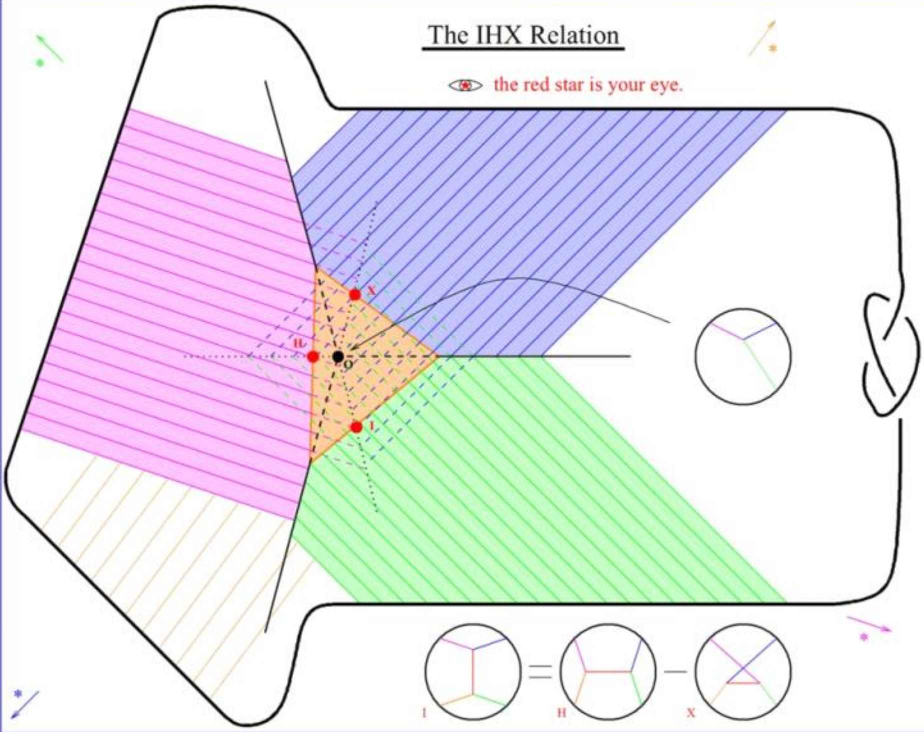
When deforming, catastrophes occur when:

A plane moves over an intersection point – Solution: Impose IHX,	An intersection line cuts through the knot – Solution: Impose STU,	The Gauss curve slides over a star – Solution: Multiply by a framing–dependent counter–term.
 (see below)	 (similar argument)	(not shown here)

Theorem. Modulo Relations, $Z(K)$ is a knot invariant!

The IHX Relation

the red star is your eye.



The Cast

in rough historical order

The Neolithic People

Carl Friedrich Gauss

~~Sophus Lie~~ remove

Edward Witten

Victor Vassiliev add Gaussian

Maxim Kontsevich

Raoul Bott

Clifford Taubes

Thang Le

Jun Murakami

 center

Tomotada Ohtsuki

put labels below images.

Cosmic Coincidences and Several Other Stories, 2

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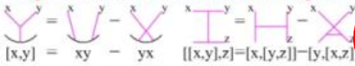
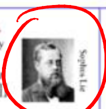
"Low Algebra" and universal formulae in Lie algebras.

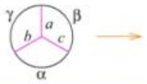




More precisely, let $\mathfrak{g} = \langle X_a \rangle$ be a Lie algebra with an orthonormal basis, and let $R = \langle v_\alpha \rangle$ be a representation. Set

$$f_{abc} := \langle [a,b], c \rangle \qquad X_a v_\beta = \sum_\beta r^\beta_{a\gamma} v_\gamma$$

and then



$$W_{\mathfrak{g},R}: \longrightarrow \sum_{abc\alpha\beta\gamma} f_{abc} r^\beta_{a\gamma} r^\gamma_{b\alpha} r^\alpha_{c\beta}$$

$W_{\mathfrak{g},R} \circ Z$ is often interesting:

$\mathfrak{g} = sl(2) \longrightarrow$ The Jones polynomial

$\mathfrak{g} = sl(N) \longrightarrow$ The HOMFLYPT polynomial

$\mathfrak{g} = so(N) \longrightarrow$ The Kauffman polynomial

Chern-Simons-Witten theory and Feynman diagrams.

$$\int_{\mathfrak{g}\text{-connections}} \mathcal{D}A\, hol_K(A) \exp\left[\frac{ik}{4\pi}\int_{\mathbb{R}^3} \operatorname{tr}\left(A \wedge dA + \frac{2}{3}A\wedge A\wedge A\right)\right]$$

$$\longrightarrow \sum_{D:\ \text{Feynman diagram}} W_{\mathfrak{g}}(D) \sum \mathcal{E}(D) \longrightarrow \sum_{D:\ \text{Feynman diagram}} D \sum \mathcal{E}(D)$$

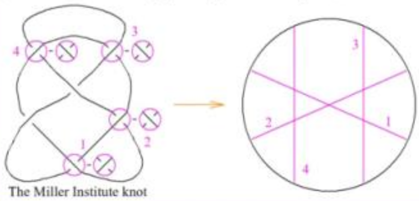
Definition. V is finite type (Vassiliev, Goussarov) if it vanishes on sufficiently large alternations as on the right

Theorem. All knot polynomials (Conway, Jones, etc.) are of finite type.

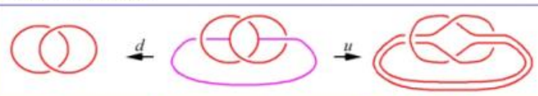
Conjecture. (Taylor's theorem) Finite type invariants separate knots.

Theorem. $Z(K)$ is a universal finite type invariant!

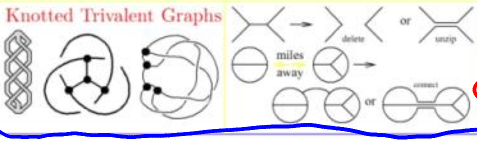
(sketch: to dance in many parties, you need many feet).

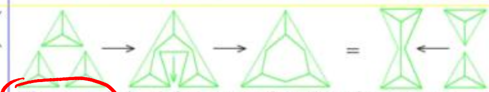


Knots are the wrong objects to study in knot theory! They are not finitely generated and they carry no interesting operations.



Knotted Trivalent Graphs





Theorem (~). A homomorphic Z is the same as a "Drinfel'd Associator".

enlarge Lie

thick line

Red, "High Algebra".

Import u→v→w (P) from UWO, perhaps transposed.

Colour code the back ground:

dark green: explained
light green: sketched
yellow: could have explained
orange: could explain, gaps remain
light red: more gaps than explanation
dark red: mystery
where's hypo?

topology, combinatorics, low algebra, high algebra, counting coincidences, conf space integrals, QFT, graph homology

The ~~u-v-w Story~~ u→v→w (&P) stories, very rough.

	u-Knots ⟶	v-Knots ⟶	w-Knots
Topology	Ordinary (usual) knotted objects in 3D — braids, knots, links, tangles, knotted graphs, etc.	Virtual knotted objects — "algebraic" knotted objects, or "not specifically embedded" knotted objects; knots drawn on a surface, modulo stabilization.	Ribbon 2D knotted objects in 4D; "flying rings". Like v, but also with "overcrossings commute".
...torics	Chord diagrams and Jacobi diagrams, modulo	Arrow diagrams and v-Jacobi diagrams,	Like v, but also with "tails commute". Only

The ~~u-v-w Story~~ u→v→w (&?) stories, very rough.

	u-Knots →	v-Knots →	w-Knots
Topology	Ordinary (usual) knotted objects in 3D — braids, knots, links, tangles, knotted graphs, etc.	Virtual knotted objects — "algebraic" knotted objects, or "not specifically embedded" knotted objects; knots drawn on a surface, modulo stabilization.	Ribbon 2D knotted objects in 4D; "flying rings". Like v, but also with "overcrossings commute".
Combinatorics	Chord diagrams and Jacobi diagrams, modulo $4T$, STU, IHX, etc.	Arrow diagrams and v-Jacobi diagrams, modulo $6T$ and various "directed" STUs and IHXs, etc.	Like v, but also with "tails commute". Only "two in one out" internal vertices.
Low Algebra	Finite dimensional metrized Lie algebras, representations, and associated spaces.	Finite dimensional Lie bi-algebras, representations, and associated spaces.	Finite dimensional co-commutative Lie bi-algebras (i.e., $\mathfrak{g} \ltimes \mathfrak{g}^*$), representations, and associated spaces.
High Algebra	The Drinfel'd theory of associators.	Likely, quantum groups and the Etingof-Kazhdan theory of quantization of Lie bi-algebras.	The Kashiwara-Vergne-Alekseev-Torossian theory of convolutions on Lie groups and Lie algebras.