Zsuzsi on 2-Categories

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2-Category:

* objects (0-cells): $A, B, \ldots$
* 1-morphisms (1-cells): $f: A \to B$.
* 2-morphisms (2-cells): $\alpha: (f: A \to B) \Rightarrow (g: A \to B)$

Vertical composition:

$$\begin{tikzcd} A \arrow[r, bend left=50] \arrow[r] \arrow[r, bend right=50] & B \end{tikzcd}$$

Horizontal composition

$$\begin{tikzcd} A \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Uparrow"] & B \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Uparrow"] & C \end{tikzcd}$$

Subject to several obvious axioms ----, the fun being the well-definedness of

$$\begin{tikzcd} \bullet \arrow[r, bend left=50] \arrow[r] \arrow[r, bend right=50] & \bullet \arrow[r, bend left=50] \arrow[r] \arrow[r, bend right=50] & \bullet \end{tikzcd}$$

What's a 2-category on one object

— that's the same as a monoidal category.

The 2-category of categories:

Objects: Categories

1-mor: Functors.

2-mor: natural transformations.

<u>Adjoints</u>

$$\begin{tikzcd} \hat{F}: \mathcal{A} \arrow[r, bend right, "\hat{F}"'] & \mathcal{B}: F \arrow[l, bend right, "F"'] \end{tikzcd}$$

left adjoint

right adjoint.

means that there is a natural isomorphisms

$$\phi: \mathrm{Hom}_{\mathcal{A}}(\hat{F}-, -) \to \mathrm{Hom}_{\mathcal{B}}(-, F-)$$

Example.

$$\begin{tikzcd} \hat{F}: \text{sets} \arrow[r, bend left, "\text{free}"] & \text{Groups}: F \arrow[l, bend left] \end{tikzcd}$$

Forgetfull

Generalise to 2-cat.

$\mathrm{Hom}_{\mathcal{A}}(\hat{F}Y, \hat{F}Y) \overset{\Phi}{\longleftrightarrow} \mathrm{Hom}_{\mathcal{B}}(Y, F\hat{F}Y)$

$\mathrm{Id} \rightsquigarrow (Y \to F\hat{F}Y)$

gives a natural tr. $\eta: 1_{\mathcal{B}} \Rightarrow F\hat{F}$ unit

$\mathrm{Hom}(FX, FX) \overset{\Phi}{\longleftrightarrow} \mathrm{Hom}(\hat{F}FX, X)$

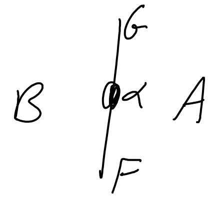
$\mathrm{Id} \rightsquigarrow (\hat{F}FX, X)$ co-unit

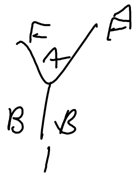
$\varepsilon: \hat{F}F \Rightarrow 1_{\mathcal{A}}$

$\hat{F} \overset{\hat{F}\eta}{\Longrightarrow} \hat{F}F\hat{F} \overset{\varepsilon\hat{F}}{\Longrightarrow} \hat{F}$ composes to $1_{\hat{F}}$

$F \overset{\eta F}{\Longrightarrow} F\hat{F}F \overset{F\varepsilon}{\Longrightarrow} F$ composes to 1_F

Alternative notation:

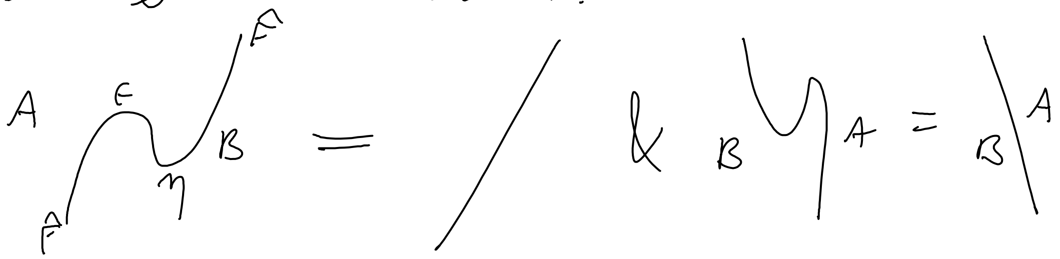
$$\begin{tikzcd} A \arrow[r, bend left, "G"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Uparrow\alpha"] & B \end{tikzcd}$$ becomes 

$\eta: 1_B \Rightarrow F\hat{F}$ becomes 

or simply 

Likewise ϵ becomes 

, so the adjointness axioms become:



<u>Prop</u> Composites of adjoints are adjoints:

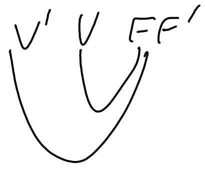
$F: B \underset{U}{\overset{F}{\rightleftarrows}} A : U \qquad F': C \overset{F'}{\rightleftarrows} B : U'$

$\eta: 1_B \Rightarrow UF \qquad \eta': 1_C \Rightarrow U'F'$

$\eta: 1_B \Rightarrow UF \qquad \eta': 1_C \Rightarrow U'F'$

$\epsilon: FU \Rightarrow 1_A \qquad \epsilon': F'U' \Rightarrow 1_B$

$$\begin{tikzcd} FF' : C \arrow[r, bend left] \arrow[r, phantom, "\perp"] & A : U'U \arrow[l, bend left, "U'U"] \end{tikzcd}$$

with $\bar{\eta}$: 

$\bar{\epsilon}$: 

Biadjoints: 

<u>Example</u> $H \subset G$

Res: G-mod $\to$ H-mod

ind: H-mod $\to$ G-mod by $M \mapsto G \otimes_H M$

One adjointness is general. If G is finite, Fröbenius reciprocity gives the other adjointness.

<u>Thm</u> The bi-adjoint $\hat{F}$ of any 1-mor F is unique up to isom.

Duals for 2-morphisms: