

# RCI Lecture as of February 6

February-06-11

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**The Hardest Math I've Ever Really Used**  
Drew Bernstein at the Royal Canadian Institute, February 2011. <http://www.math.utoronto.edu/~dbernstein/Talks/RCI-110211W>

**Abstract:** What's the hardest math I've ever used in real life? Myself, obviously — not by using a calculator or a GPS device that somebody else designed? And in "real life" — not while studying or teaching mathematics? One solution and subtraction: daily, adding up bills or calculating change. I use percentages when, through necessity, it's just "add 10 percent". I seldom use multiplication and division: when I buy in bulk, or when I need to know how many tiles I need to replace on kitchen floor. I've used power twice in my life, doing calculations related to mortgages. I've used a tiny bit of geometry and algebra for a tiny bit of research-related computer graphics I've played with. And for a long time, that was all. In my talk I will tell you how recently a math topic discovered only in the 1980s made a real and modest appearance in my non-academical life. There are many books devoted to that topic and a lot of active research. Yet for all I know, nobody ever needed the actual formulas for such a simple reason: nobody.

How do I talk about the notion of metric curves, and the fastest way to go from A to B subject to driving speed limits that depend on the roads, and the "happy square principle" which is at the heart of the best action principle which is itself at the heart of all of modern physics, and finally, about that fringe discovery of Jean-Benoit and Nikolai Ivanovich Lobachevsky, that the least action of particles of the ancient Greeks need not actually be true.

**...or an Art Historian...**  
**...or an Anthropologist**  
**...or an Art Historian...**  
**...or an Anthropologist**

**Find the best-kept path to go from Mona's left eye to Mona's right eye in 3D!**  
Alternatively, fix your line-of-sight, and find the fastest path to do the same. For fixed line, our curves occur at a speed proportional to the distance from the target plane.

**Fastest path with speed limit**

**Person's Principle**

**The Hardest Math I've Ever Really Used - Page 2**

**The Happy Square Principle**  
A Square is happy if both its sides are happy.

**Happy counter-rotating Squares above the Moon Plane**

**The Lobachevsky Space**

**Parameterization**

**The Actual Code**

**Unhappy Squares**

**Happy Squares**

**The Moon Plane**

**Parameterization**

**The Actual Code**

**The Hardest Math I've Ever Really Used**

Dror Bar-Natan at the Royal Canadian Institute, February 2011, <http://www.math.toronto.edu/~drorbn/Talks/RCI-110213/>

**Abstract.** What's the hardest math I've ever used in real life? Me, myself, directly - not by using a cellphone or a GPS device that somebody else designed? And in "real life" — not while studying or teaching mathematics?

I use addition and subtraction daily, adding up bills or calculating change. I use percentages often, though mostly it is just "add 15 percents". I seldom use multiplication and division: when I buy in bulk, or when I need to know how many tiles I need to replace my kitchen floor. I've used powers twice in my life, doing calculations related to mortgages. I've used a tiny bit of geometry and algebra for a tiny bit of non-math-related computer graphics I've played with. And for a long time, that was all. In my talk I will tell you how recently a math topic discovered only in the 1800s made a brief and modest appearance in my non-mathematical life. There are many books devoted to that topic and a lot of active research. Yet for all I know, nobody ever needed the actual formulas for such a simple reason before.

Hence we'll talk about the motion of movie cameras, and the fastest way to go from A to B subject to driving speed limits that depend on the locale, and the "happy segway principle" which is a the heart of the least action principle which in itself is at the heart of all of modern physics, and finally, about that funny discovery of Janos Bolyai's and Nikolai Ivanovich Lobachevsky's, that the famed axiom of parallels of the ancient Greeks need not actually be true.

The slide contains several sections of mathematical text and diagrams:

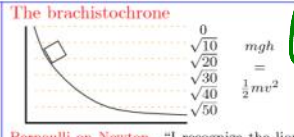
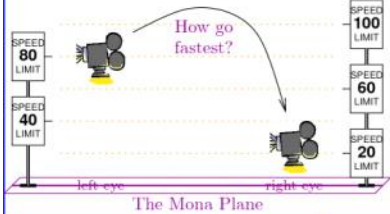
- The Problem:** Let  $G = \langle g_1, \dots, g_n \rangle$  be a subgroup of  $S_n$  with  $n = O(100)$ . Return you do, understand  $G$ .
- Compute  $g_i$ :** Given  $n \in S_n$ , decide if  $n \in G$ .
- Write  $n \in G$  in terms of  $g_1, \dots, g_n$ :** Produce random elements of  $G$ .
- Commutative Analog:** Let  $V = \langle v_1, \dots, v_n \rangle$  be a subspace of  $\mathbb{R}^n$ . Return you do, understand  $V$ .
- Subspace:** Gaussian Elimination. Prepare a matrix  $A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}$ .
- Non-Commutative Gaussian Elimination:** Prepare a matrix  $A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}$ .
- Space for  $n \sigma_i \in S_n$  of the form  $(1, 2, \dots, i-1, j, i, i+1, \dots, n)$ :** So  $\sigma_i$  from  $1, \dots, i-1, j, i, i+1, \dots, n$  — make "the proof"  $i$  to  $j$  and guess wild afterwards, and  $\sigma_i^2$  "does steeper".
- Find  $g_1, \dots, g_n$  in order:** To find a non-identity  $\sigma$ , find its pre-ordered position  $i$  and let  $j = \sigma(i)$ .
- If box  $(i, j)$  is empty, put a stone.**
- If box  $(i, j)$  contains  $\sigma_{ij}$ :** Feed  $\sigma' = \sigma \sigma_{ij}^{-1}$ .
- The Trick:** When done, for every occupied  $(i, j)$  and  $(k, l)$ , feed  $\sigma_{ij} \sigma_{kl}$ . Repeat until the table stops changing.
- Claim:** The process stops in our lifetimes, after at most  $O(n^2)$  operations. Call the resulting table  $T$ .
- Claim:** Anything fed in  $T$  is a nontrivial element in  $G$ .
- If  $\sigma$  was fed  $\Rightarrow f \in M := \langle \sigma_{12}, \sigma_{23}, \dots, \sigma_{n-1,n} \rangle$ ,  $\forall j \geq i \geq 1, \sigma_{ij} \in T$ .**
- Homework Problem 1:** Can you do better?
- Homework Problem 2:** Can you do categories (groupoids)?



The Canton Tower in Guangzhou during the 2010 Asian Games Opening Ceremony, as photographed by Colin Zh...

Its hyperbolic structure, first explored by Russian engineer Vladimir Shukhov in the 1880s, is known for its strength,

**Goal.** Find the least-blur path to go from Mona's left eye to Mona's right eye in fixed time. Alternatively, fix your blur-tolerance, and find the fastest path to do the same. For fixed blur, our camera moves at a speed proportional to its distance from the image plane:



Bernoulli on Newton. "I recognize the lion by his paw".



The principle of least action

structural simplicity, and its relationship with hyperbolic geometry and with Einstein's theory of relativity.



this would be better on the other side.






```

ParametricPlot3D[
  {Sin[u] Cos[v],
   Sin[u] Sin[v],
   Cos[u]},
  {u, 0, Pi}, {v, 0, 2*Pi}]


```



```


ParametricPlot3D[
  {Sech[u] Cos[v],
   Sech[u] Sin[v],
   u - Tanh[u]},
  {u, 0, Pi}, {v, 0, 2*Pi}]

```



### The Happy Segway Principle

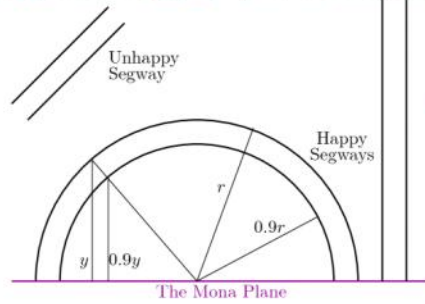
A Segway is happy iff both its wheels are





happy unhappy

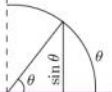
Happy camera-carrying Segways above the Mona Plane. The Lobachevsky Space



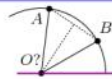
Parametrization

$$\theta'(t) = \sin \theta(t)$$

$$\Downarrow$$

$$\theta = 2 \arctan e^t$$


Rotations, translations and some basic geometry also occur



### The Actual Code

```

x = p1.x-p2.x; y = p1.y-p2.y;
d1 = p1.d; d2 = p2.d;
norm = sqrt(x*x + y*y);
a = x/norm; b = y/norm;
x1p = a*x + b*y;
x0 = (x1p + (d1*d1-d2*d2)/x1p)/2;
r = sqrt((x1p-x0)*(x1p-x0)+d1*d1);
x1pp = (x1p-x0)/r; x2pp = -x0/r;
theta1 = acos(x1pp);
theta2 = acos(x2pp);
t1 = log(tan(theta1/2));
t2 = log(tan(theta2/2));
t3 = t1 + a*(t2-t1);
theta3 = 2*atan(exp(t3));
x3pp = cos(theta3);
y3pp = sin(theta3);
x3p = x0 + r*x3pp;
p3.d = r*d3pp;
p3.x = p2.x + a*x3p;
p3.y = p2.y + b*x3p;

```

Ops used: +, -, \*, /, sqrt, cos, sin, tan, arccos, arctan, log, exp.