BEER-4 by Peter Lee, November 2

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9:30 AM

P_n = Convex hull of $S_n \subset \mathbb{R}^n$

Claim 1. No vertex is in the hull of any others.

2. The faces of P_n correspond to ordered partitions of $[n] = \{1, \dots, n\}$

Def If $(a_1 \dots, a_n)$ is a permutation of $[n]$, define $(a_1, \dots a_{i_1 - 1})(a_{i_1}, \dots a_{i_2 - 1}) \dots (a_{i_k}, \dots a_n)$ to be the convex hull of the permutations in which

$\forall \ell \quad \{a_{i_\ell}, \dots, a_{i_{\ell+1}-1}\} = \{i_\ell, \dots, i_{\ell+1}-1\}$

e.g. $(13)(24) = \text{Conv}\begin{Bmatrix} (1,3,2,4) & (2,3,1,4) \\ (1,4,2,3) & (2,4,1,3) \end{Bmatrix}$

$= \quad (1)(3)(2)(4) — (3)(1)(2)(4)$

$\vdots$

Let $V = (a_1)(a_2) \dots (a_n) = a_1 \dots a_n$

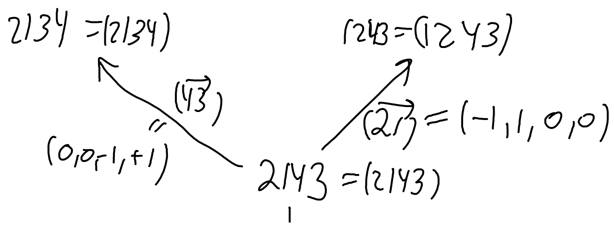
Def $(\overrightarrow{a_i a_{i+1}})$ is the oriented segment from V to $a_1 \dots a_{i+1}\, a_i \dots a_n$

"the basic inward vectors at V" ("biv")

as a vector, $(\overrightarrow{a_i, a_{i+1}})$ is

$$(0, \dots 0, \underset{a_i}{+1}, \dots \underset{a_{i+1}}{-1}, \dots 0)$$

Example



$\downarrow (\overrightarrow{14}) = (+1, 0, 0, -1)$

$2413 = (3142)$

<u>Lemma</u> The vectors $(\overrightarrow{a_i a_j})$ for $1 \leq i < j \leq n$,

$$= (0 \ldots \underset{a_i}{+1}, \ldots \underset{a_j}{-1}, \ldots) = e_{a_i} - e_{a_j}$$

are in the non-negative integer span of the basic inward vectors.

<u>Lemma</u> All vertices in P^n are in the non-negative integer span of the biv's at V.

<u>Proof</u> On board.

<u>Corollary</u> No vertex of P_n is in the convex hull of others.

<u>Note</u> The biv make a basis for $\mathbb{R}^{n-1} \cong A_n = \{(x) \in \mathbb{R}^n : \sum x_i = \binom{n}{2}\}$

<u>Note</u> Any $(n-2)$ biv at V define a hyperplane H in A_n s.t. P_n lies entirely on one side of H (corresponding to the direction of the remaining biv).

<u>Def</u> Let H_i^V be the hyperplane given by all biv's at V other than $(\overrightarrow{a_i a_{i+1}})$, and H_i^{V+} the corresponding non-negative span.

<u>Note</u> $H_i^V \cap P_n$ is on the bndry of P_n

<u>Claim</u> $H_i^{V+} \cap P_n = (a_1 \ldots a_i)(a_{i+1} \ldots a_n)$

<u>PF</u> on board.

<u>Def</u> Write $H^V_{i_1 \ldots i_k}$ for the subspace generated by the biv at V other than $(\overrightarrow{a_i a_{i+1}})$

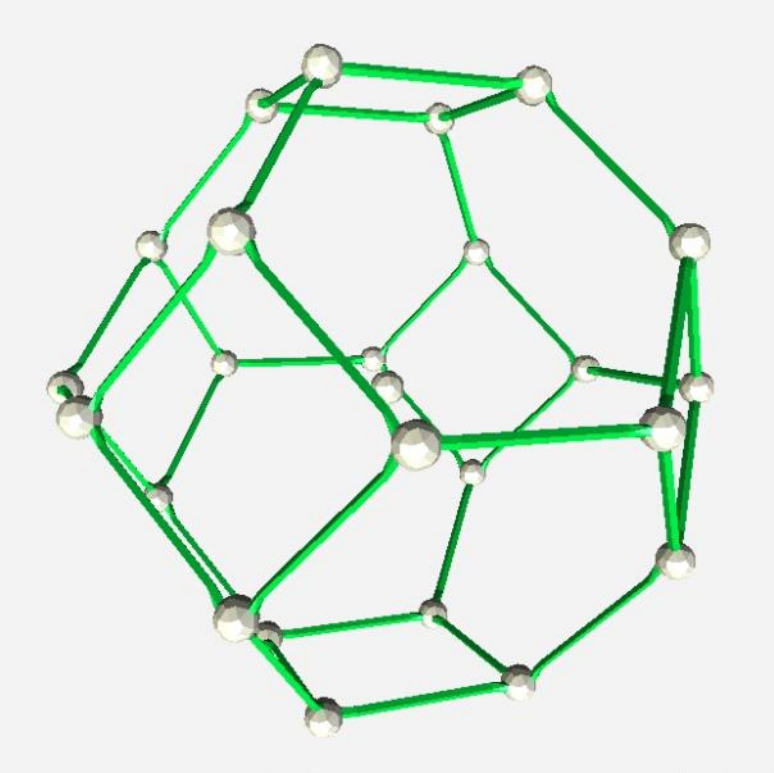
the biv at V other than $(\overrightarrow{a_{i_l} a_{i_l+1}})$

<u>claim</u> $\bigcap_{l=1..k} H_{i_k}^{\vee} = H_{i_1 \dots i_k}^{\vee}$

<u>claim</u> $H_{i_1 \dots i_k}^{\vee} \cap P_n = (a_1 \dots a_{i_1})(a_{i_1+1} \dots a_{i_2}) \dots (a_{i_k+1} \dots a_n)$

<u>Pf</u> (Same as the previous omitted proof).

... a discussion of the relationship with weight diagrams....



This image shows a weight diagram for the representation whose highest weight is "(1,1,1)," that is, the
sum of the three fundamental weights. What we have here is just the orbit of the highest weight under the
Weyl group (24 elements). These form the vertices of the "weight polyhedron" for this representation.

Pasted from <http://www.nd.edu/~bhall/book/a3wtdiag1.html>