Ram Murty@Colloq: Transcendental Numbers and L-functions

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General question: Given an aut. rep. $\pi$, what can we say about $L(1,\pi)$ or more generally, about $L(k,\pi)$, $k\in\mathbb{Z}$

Dirichlet: Given $\chi: (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}^*$

set
$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

Euler: $\zeta(2k)\in \pi^{2k}\mathbb{Q}^*$, but little is known about $\zeta(2k+1)$.

Similarly,

$L(2k,\chi)$ when $\chi(-1)=1$

& $L(2k+1,\chi)$ when $\chi(-1)=-1$ are controlled, but transcendentality of the rest is open.

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Let $K$ be an imaginary quadratic field, let $\chi$ be a character of the F-ideal class group of $\mathcal{O}_K$. We wish to study $L(1,\chi)$

<u>Definitions</u> Given an algebraic field $K$,

$\mathcal{O}_K$: The ring of integers in $K$.

It is a Dedekind domain - integral domain with a unique factorization property for ideals.

Given an ideal $F$ in $\mathcal{O}_K$, let $\mathfrak{a}\sim\mathfrak{b}$ for ideals $\mathfrak{a}$ & $\mathfrak{b}$, if $(\alpha)\mathfrak{a}=(\beta)\mathfrak{b}$ for some $\alpha,\beta$ with $\alpha-\beta\in F$.

Call the group of equiv. classes $Cl(F)$

For $\chi\in\widehat{Cl(F)}$. what can we say about

$L(1,\chi)$?

Theorem 1 1. If $K$ is imaginary quadratic & $F=(1)$, Let $\chi\neq 1$. Then $L(1,\chi)/\pi$ is a $\overline{\mathbb{Q}}$-lin. comb. of logs of alg. numbers. (hence it is transcendental)

2. The $L(1,\chi)$'s are lin. indep. over $\overline{\mathbb{Q}}$.

Theorem 2 With $K$ as before & $F\neq(1)$, then the $L(1,\chi)$'s are lin. indep $/\overline{\mathbb{Q}}$ and ... same...