

Non Commutative Gaussian Elimination - Program 2

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Amended from a similar notebook by Dror Bar-Natan and Itai Bar-Natan. The original version is at <http://www.math.toronto.edu/~drorbn/Misc/SchreierSimsRubik/>.

Pensieve Header: NCGE Program 2 - replacing tricks with better ones when possible, non-recursively.
The results are acceptable.

The Cube

The Generating Permutations

```

n = 54; $RecursionLimit = 2^16;
Generators = {
  M[{18, 27, 36, 4, 5, 6, 7, 8, 9, 3, 11, 12, 13, 14, 15, 16, 17,
    45, 2, 20, 21, 22, 23, 24, 25, 26, 44, 1, 29, 30, 31, 32, 33, 34, 35, 43,
    37, 38, 39, 40, 41, 42, 10, 19, 28, 52, 49, 46, 53, 50, 47, 54, 51, 48},
    {BottomFace}, 1],
  M[{1, 2, 3, 4, 5, 6, 16, 25, 34, 10, 11, 9, 15, 24, 33, 39, 17,
    18, 19, 20, 8, 14, 23, 32, 38, 26, 27, 28, 29, 7, 13, 22, 31, 37, 35, 36,
    12, 21, 30, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54},
    {TopFace}, 1],
  M[{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17,
    18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 31, 32, 33, 34, 35, 36, 48, 47, 46,
    39, 42, 45, 38, 41, 44, 37, 40, 43, 30, 29, 28, 49, 50, 51, 52, 53, 54},
    {FrontFace}, 1],
  M[{3, 6, 9, 2, 5, 8, 1, 4, 7, 54, 53, 52, 10, 11, 12, 13, 14,
    15, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36,
    37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 18, 17, 16},
    {BackFace}, 1],
  M[{13, 2, 3, 22, 5, 6, 31, 8, 9, 12, 21, 30, 37, 14, 15, 16,
    17, 18, 11, 20, 29, 40, 23, 24, 25, 26, 27, 10, 19, 28, 43, 32, 33, 34, 35,
    36, 46, 38, 39, 49, 41, 42, 52, 44, 45, 1, 47, 48, 4, 50, 51, 7, 53, 54},
    {LeftFace}, 1],
  M[{1, 2, 48, 4, 5, 51, 7, 8, 54, 10, 11, 12, 13, 14, 3, 18, 27,
    36, 19, 20, 21, 22, 23, 6, 17, 26, 35, 28, 29, 30, 31, 32, 9, 16, 25, 34,
    37, 38, 15, 40, 41, 24, 43, 44, 33, 46, 47, 39, 49, 50, 42, 52, 53, 45},
    {RightFace}, 1]
};

```

Program 2

```

Clear[s, M, T]; TC = 0;
M /: M[a1_, {w1___}, m1_] ** M[a2_, {w2___}, m2_] := M[a1[[a2]], {w1, w2}, m1 + m2];
M /: Inverse[M[a_, w_, m_]] := M[Ordering[a], -Reverse[w], m];
Feed[M[Range[n], ___]] := Null;
Feed[M[a_, {w___}, m_]] := Module[
  {i, j, sij, k, l, skl},
  For[i = 1, a[[i]] == i, ++i]; j = a[[i]];
  If[Head[sij = s[i, j]] === Integer,
    (* then *) If[m ≥ T[sij][[3]],
      Feed[ReplacePart[Inverse[T[sij]] ** M[a, {w}, m], {-sij, w}, 2]],
      T[s[i, j] = ++TC] = M[a, {w}, m];
      Feed[ReplacePart[Inverse[M[a, {w}, m]] ** T[sij], -{w, -sij}, 2]]
    ],
    (* else *) T[s[i, j] = ++TC] = M[a, {w}, m];
  Do[
    If[Head[skl = s[k, l]] == Integer,
      Feed[ReplacePart[T[sij] ** T[skl], {sij, skl}, 2]];
      Feed[ReplacePart[T[skl] ** T[sij], {skl, sij}, 2]]
    ],
    {k, n}, {l, n}
  ]
];
Images[i_] := Prepend[Select[Range[n], Head[s[i, #]] === Integer &], i]

```

The Order of the Group

```

Timing[
  (Feed[#]; Product[Length[Images[i]], {i, n}]) & /@ Generators
]
{111.384, {4, 16, 159 993 501 696 000,
  21 119 142 223 872 000, 43 252 003 274 489 856 000, 43 252 003 274 489 856 000}}

```

The Worst Case Scenario

```

MoveCount[i_, i_] := 0;
MoveCount[i_, j_] := T[s[i, j]][[3]];
Sum[Max[MoveCount[i, #] & /@ Images[i]], {i, n}]
3089

Timing[
  (Feed[#]; Product[Length[Images[i]], {i, n}]) & /@ (Inverse /@ Generators)
]
{0.078, {43 252 003 274 489 856 000, 43 252 003 274 489 856 000, 43 252 003 274 489 856 000,
  43 252 003 274 489 856 000, 43 252 003 274 489 856 000, 43 252 003 274 489 856 000}}

```

```

MoveCount[i_, j_] := 0;
MoveCount[i_, j_] := T[s[i, j]][[3]];
Sum[Max[MoveCount[i, #] & /@ Images[i]], {i, n}]
3089
    
```

For the Patient : A Recipe for Solving the Cube

```

ColumnForm /@ {
  Reap[
    Do[If[Head[s[i, j]] === Integer, Sow[S[i, j] → s[i, j]]], {i, n-1}, {j, i+1, n}]
  ][[2, 1]],
  (# → Rest[T[#]]) & /@ Range[1, TC, 100]
}
{ S[1, 3] → 4659 , 1 → M[{BottomFace}, 1]
  S[1, 7] → 7667 101 → M[{-56, -88, -53, -50, -24, -26, -19, -66, 100, 66}, 1511879745]
  S[1, 9] → 7543 201 → M[{7, 133}, 236364402984]
  S[1, 10] → 4389 301 → M[{122, 40}, 44864800173]
  S[1, 12] → 4966 401 → M[{-117, -17, -82, 117, 51}, 14261147372]
  S[1, 13] → 4993 501 → M[{-10, 113, 13}, 6835104295]
  S[1, 15] → 4338 601 → M[{-80, -111, 95, 111}, 3677447834]
  S[1, 16] → 4110 701 → M[{-21, -72, -108, 4, 108}, 3172836079]
  S[1, 18] → 1 801 → M[{-48, -71, 102, 71}, 1547721149]
  S[1, 28] → 7664 901 → M[{-75, -883, -96, -5, 863}, 1564819171]
  S[1, 30] → 4552 1001 → M[{-70, 92, 70}, 313563956]
  S[1, 31] → 4537 1101 → M[{-54, -21, -39, -19, -9, 87, 18}, 240571069]
  S[1, 33] → 4333 1201 → M[{82, 74}, 187966394]
  S[1, 34] → 4540 1301 → M[{-21, -78, -65, -25, -18, 78, 74}, 107266026]
  S[1, 36] → 9 1401 → M[{-17, -19, -13, 77, 40}, 35774145]
  S[1, 37] → 4550 1501 → M[{21, 76}, 29803514]
  S[1, 39] → 4544 1601 → M[{-44, -25, -2, 73, 35}, 17912231]
  S[1, 43] → 4551 1701 → M[{-7, -72, 1671}, 23873706]
  S[1, 45] → 2 1801 → M[{4, 70}, 5970575]
  S[1, 46] → 7668 1901 → M[{-29, -1874, -26, 1790, 65}, 14911574]
  S[1, 48] → 4543 2001 → M[{61, 1949}, 4315118]
  S[1, 52] → 4391 2101 → M[{58, 2073}, 1580762]
  S[1, 54] → 4459 2201 → M[{55, 6}, 275578]
  S[2, 4] → 7570 2301 → M[{53, 2300}, 308951]
  S[2, 6] → 4662 2401 → M[{-27, -6, -23, -51, 2339, 51}, 275193]
  S[2, 8] → 7562 2501 → M[{-2386, -2482, -2420, -2347, -49, -2346, 2340}, 1360539]
  S[2, 11] → 7541 2601 → M[{-6, 47, 2595}, 152854]
  S[2, 14] → 7416 2701 → M[{-30, 43, 20, -45, -20, 2687}, 72030]
  S[2, 17] → 7412 2801 → M[{-42, 2783, 44}, 136373]
  S[2, 19] → 4379 2901 → M[{8, 41}, 16474]
  S[2, 21] → 4412 3001 → M[{-2999, 2666, 37}, 33418]
  S[2, 22] → 7419 3101 → M[{3100, 35}, 40737]
  S[2, 24] → 7564 3201 → M[{-7, 31, 7}, 4886]
  S[2, 25] → 4381 3301 → M[{-10, 27, 10}, 1242]
  S[2, 27] → 11 3401 → M[{-3378, 26, -3323, -26, 3365}, 37975]
  S[2, 29] → 7558 3501 → M[{3469, 21, 8, -3437, -23, 3292}, 2584]
  S[2, 32] → 7414 3601 → M[{-3482, -3505, 3600, 21}, 4864]
  S[2, 35] → 7542 3701 → M[{3, 18}, 55]
  S[2, 38] → 7669 3801 → M[{-6, -17, 3736, 17}, 212]
  S[2, 40] → 4660 3901 → M[{-3819, -16, 3881, 16}, 162]
  S[2, 42] → 4538 4001 → M[{-3993, -3885, -3807, -15, 3993, 15}, 4792]
  S[2, 44] → 4994 4101 → M[{-6, -13, 3904, 13}, 104]
  S[2, 47] → 7548 4201 → M[{-1170, 3805, 12}, 2021]
    
```

$S[2, 47] \rightarrow 7540$
 $S[2, 49] \rightarrow 7561$
 $S[2, 51] \rightarrow 7567$
 $S[2, 53] \rightarrow 7492$
 $S[3, 7] \rightarrow 4973$
 $S[3, 9] \rightarrow 4435$
 $S[3, 12] \rightarrow 4426$
 $S[3, 13] \rightarrow 4190$
 $S[3, 15] \rightarrow 7672$
 $S[3, 16] \rightarrow 4521$
 $S[3, 18] \rightarrow 4077$
 $S[3, 28] \rightarrow 4533$
 $S[3, 30] \rightarrow 4496$
 $S[3, 31] \rightarrow 4566$
 $S[3, 33] \rightarrow 4569$
 $S[3, 34] \rightarrow 4571$
 $S[3, 36] \rightarrow 4573$
 $S[3, 37] \rightarrow 4514$
 $S[3, 39] \rightarrow 4575$
 $S[3, 43] \rightarrow 4577$
 $S[3, 45] \rightarrow 4576$
 $S[3, 46] \rightarrow 4574$
 $S[3, 48] \rightarrow 7661$
 $S[3, 54] \rightarrow 4488$
 $S[4, 6] \rightarrow 4981$
 $S[4, 8] \rightarrow 4949$
 $S[4, 11] \rightarrow 7603$
 $S[4, 14] \rightarrow 7635$
 $S[4, 17] \rightarrow 7335$
 $S[4, 19] \rightarrow 4968$
 $S[4, 21] \rightarrow 7536$
 $S[4, 22] \rightarrow 7574$
 $S[4, 24] \rightarrow 7647$
 $S[4, 25] \rightarrow 4988$
 $S[4, 27] \rightarrow 4661$
 $S[4, 29] \rightarrow 4995$
 $S[4, 32] \rightarrow 7656$
 $S[4, 35] \rightarrow 7657$
 $S[4, 38] \rightarrow 4977$
 $S[4, 40] \rightarrow 4969$
 $S[4, 42] \rightarrow 4978$
 $S[4, 44] \rightarrow 7610$
 $S[4, 47] \rightarrow 7658$
 $S[4, 49] \rightarrow 7546$
 $S[4, 51] \rightarrow 7670$
 $S[6, 8] \rightarrow 4996$
 $S[6, 14] \rightarrow 7469$
 $S[6, 17] \rightarrow 7460$
 $S[6, 19] \rightarrow 7537$
 $S[6, 21] \rightarrow 4984$
 $S[6, 22] \rightarrow 7471$
 $S[6, 24] \rightarrow 7662$
 $S[6, 25] \rightarrow 4982$
 $S[6, 27] \rightarrow 4958$
 $S[6, 29] \rightarrow 7476$
 $S[6, 32] \rightarrow 7467$
 $S[6, 35] \rightarrow 7596$
 $S[6, 38] \rightarrow 4985$
 $S[6, 40] \rightarrow 4934$
 $S[6, 42] \rightarrow 4972$
 $S[6, 44] \rightarrow 4986$
 $4201 \rightarrow M[{-4179, 5009, 12}, 292]$
 $4301 \rightarrow M[{-4080, -4214, 4149, 10}, 60]$
 $4401 \rightarrow M[{-4381, -4397, 4379, 9}, 58]$
 $4501 \rightarrow M[8, -6, -8, 4403}, 38]$
 $4601 \rightarrow M[7, 4593}, 28]$
 $4701 \rightarrow M[{-4435, -4660, -4543, 4675, 4543}, 1403]$
 $4801 \rightarrow M[4585, 4669}, 569]$
 $4901 \rightarrow M[{-4190, -4412, -4110, 4664, 4331}, 372]$
 $5001 \rightarrow M[{-4659, 5000, 4659}, 179]$
 $5101 \rightarrow M[{-4435, -4994, -4537, 5064, 4338}, 923965]$
 $5201 \rightarrow M[{-4500, -5013, -4426, 5056, 4435}, 508475]$
 $5301 \rightarrow M[{-4562, 5052, 4615}, 333081]$
 $5401 \rightarrow M[{-4574, -11, -4552, 5047, 4550}, 191500]$
 $5501 \rightarrow M[{-4543, 5044, 4543}, 154308]$
 $5601 \rightarrow M[{-4499, -4, -11, -4333, 5041, 4543}, 143199]$
 $5701 \rightarrow M[{-4949, 5009, 5039}, 141955]$
 $5801 \rightarrow M[4626, 5034}, 60375]$
 $5901 \rightarrow M[4662, 5029}, 20122]$
 $6001 \rightarrow M[{-4539, -4, -4533, -4381, -2, 5026, 4552}, 9707]$
 $6101 \rightarrow M[4616, 5025}, 9630]$
 $6201 \rightarrow M[{-4992, -4993, 5021, 4993}, 9459]$
 $6301 \rightarrow M[{-4962, -4997, -4571, -4412, -4338, 5019, 4338}, 4725]$
 $6401 \rightarrow M[5018, 5006}, 4975]$
 $6501 \rightarrow M[5016, 5000}, 3939]$
 $6601 \rightarrow M[{-4623, -4539, -4498, -4958, -4988, 5003, 5015}, 2598]$
 $6701 \rightarrow M[{-4599, 5012, 4498}, 1910]$
 $6801 \rightarrow M[{-4479, -4611, -4998, -4500, -4996, -4566, -5000, 4379, 5010}, 14]$
 $6901 \rightarrow M[{-4562, -4592, -4190, 5008, 4190}, 1019]$
 $7001 \rightarrow M[4587, 5007}, 941]$
 $7101 \rightarrow M[{-4314, -4629, -4626, -4547, -4611, -4586, -4498, -4996, -4967, 7201} \rightarrow M[{-4592, -4571, 5003, 4576}, 549]$
 $7301 \rightarrow M[{-4185, -4660, -4552, 5001, 4551}, 213]$
 $7401 \rightarrow M[4994, 5000}, 184]$
 $7501 \rightarrow M[{-4583, -4573, 4998, 4185}, 199]$
 $7601 \rightarrow M[4620, 4500, 7574, -4995, -7586, 7550}, 231]$

S[6, 47] → 7597
S[6, 49] → 7616
S[6, 51] → 7673
S[7, 9] → 7634
S[7, 12] → 7659
S[7, 13] → 4503
S[7, 15] → 4512
S[7, 16] → 4
S[7, 28] → 4599
S[7, 30] → 7665
S[7, 31] → 4498
S[7, 33] → 4583
S[7, 34] → 4567
S[7, 36] → 4585
S[7, 37] → 4509
S[7, 39] → 5
S[7, 43] → 4596
S[7, 45] → 4592
S[7, 46] → 4588
S[7, 48] → 4500
S[8, 14] → 7517
S[8, 19] → 4442
S[8, 21] → 7627
S[8, 22] → 7571
S[8, 24] → 7605
S[8, 25] → 4597
S[8, 27] → 4586
S[8, 29] → 7671
S[8, 32] → 7615
S[8, 35] → 7560
S[8, 38] → 4601
S[8, 40] → 4593
S[8, 42] → 4505
S[8, 44] → 4602
S[8, 47] → 7653
S[8, 49] → 7617
S[8, 51] → 7613
S[9, 15] → 4961
S[9, 16] → 7637
S[9, 28] → 4616
S[9, 30] → 4615
S[9, 31] → 4962
S[9, 33] → 4499
S[9, 34] → 4562
S[9, 36] → 4605
S[9, 37] → 4561
S[9, 39] → 4611
S[9, 43] → 4613
S[9, 45] → 4474
S[9, 46] → 4609
S[9, 48] → 4578
S[19, 21] → 4479
S[19, 22] → 7606
S[19, 24] → 7539
S[19, 25] → 4290
S[19, 27] → 4495
S[19, 29] → 7611
S[19, 32] → 7575
S[19, 35] → 7554
S[19, 38] → 4620

$S[19, 40] \rightarrow 4623$
 $S[19, 42] \rightarrow 4618$
 $S[19, 44] \rightarrow 4547$
 $S[19, 47] \rightarrow 7580$
 $S[19, 49] \rightarrow 7583$
 $S[19, 51] \rightarrow 7591$
 $S[21, 22] \rightarrow 7599$
 $S[21, 24] \rightarrow 7663$
 $S[21, 25] \rightarrow 4471$
 $S[21, 27] \rightarrow 4603$
 $S[21, 29] \rightarrow 7649$
 $S[21, 32] \rightarrow 7652$
 $S[21, 35] \rightarrow 7535$
 $S[21, 38] \rightarrow 4612$
 $S[21, 40] \rightarrow 4627$
 $S[21, 42] \rightarrow 4587$
 $S[21, 44] \rightarrow 4626$
 $S[21, 47] \rightarrow 7600$
 $S[21, 51] \rightarrow 7565$
 $S[24, 25] \rightarrow 7643$
 $S[24, 27] \rightarrow 7619$
 $S[24, 29] \rightarrow 4631$
 $S[24, 32] \rightarrow 4610$
 $S[24, 35] \rightarrow 4629$
 $S[24, 38] \rightarrow 7608$
 $S[24, 40] \rightarrow 7628$
 $S[24, 42] \rightarrow 7609$
 $S[24, 44] \rightarrow 7625$
 $S[24, 47] \rightarrow 4630$
 $S[24, 51] \rightarrow 4527$
 $S[27, 29] \rightarrow 7641$
 $S[27, 32] \rightarrow 7631$
 $S[27, 35] \rightarrow 7650$
 $S[27, 38] \rightarrow 4314$
 $S[27, 40] \rightarrow 4633$
 $S[27, 42] \rightarrow 4624$
 $S[27, 44] \rightarrow 4460$
 $S[27, 47] \rightarrow 7569$
 $S[27, 51] \rightarrow 7624$
 $S[28, 30] \rightarrow 4654$
 $S[28, 31] \rightarrow 7$
 $S[28, 33] \rightarrow 4607$
 $S[28, 34] \rightarrow 4635$
 $S[28, 36] \rightarrow 4570$
 $S[28, 37] \rightarrow 4650$
 $S[28, 39] \rightarrow 4377$
 $S[28, 43] \rightarrow 4639$
 $S[28, 45] \rightarrow 4646$
 $S[28, 46] \rightarrow 4643$
 $S[28, 48] \rightarrow 7666$
 $S[29, 32] \rightarrow 4641$
 $S[29, 35] \rightarrow 4644$
 $S[29, 38] \rightarrow 7645$
 $S[29, 40] \rightarrow 7585$
 $S[29, 42] \rightarrow 7592$
 $S[29, 44] \rightarrow 7646$
 $S[29, 47] \rightarrow 4657$
 $S[30, 31] \rightarrow 4653$
 $S[30, 33] \rightarrow 4651$
 $S[30, 34] \rightarrow 4638$
 $S[30, 36] \rightarrow 4562$

$S[30, 30] \rightarrow 4303$
 $S[30, 37] \rightarrow 4558$
 $S[30, 39] \rightarrow 4642$
 $S[30, 45] \rightarrow 4252$
 $S[30, 48] \rightarrow 4545$
 $S[32, 35] \rightarrow 4655$
 $S[32, 38] \rightarrow 7654$
 $S[32, 42] \rightarrow 7594$
 $S[32, 44] \rightarrow 7660$
 $S[32, 47] \rightarrow 4658$
 $S[33, 34] \rightarrow 4494$
 $S[33, 36] \rightarrow 7655$
 $S[33, 39] \rightarrow 4649$
 $S[33, 45] \rightarrow 7636$
 $S[33, 48] \rightarrow 4581$
 $S[35, 42] \rightarrow 7632$