Trieste Handout as of May 16

May-17-09
3:50 AM

Convolutions on Lie Groups and Lie Algebras and Ribbon 2–Knots

Dror Bar–Natan, Trieste May 2009, http://www.math.toronto.edu/~drorbn/Talks/Trieste-0905

Disclaimer: Rough edges remain!

"God created the knots, all else in topology is the work of mortals." Leopold Kronecker (modified)

Convolutions statement. Convolutions of invariant functions on a Lie group agree with convolutions of invariant functions on its Lie algebra. More accurately, let G be a finite dimensional Lie group and let $\mathfrak{g}$ be its Lie algebra, let $j:\mathfrak{g}\to\mathbb{R}$ be the Jacobian of the exponential map $\exp:\mathfrak{g}\to G$, and let $\Phi:\mathrm{Fun}(G)\to\mathrm{Fun}(\mathfrak{g})$ be given by $\Phi(f)(x):=j^{1/2}(x)f(\exp x)$. Then if $f,g\in\mathrm{Fun}(G)$ are Ad-invariant and supported near the identity, then

$$\Phi(f)\star\Phi(g)=\Phi(f\star g).$$

Group-Ring statement. There exists $\omega^2\in\mathrm{Fun}(\mathfrak{g})^G$ so that for every $\phi,\psi\in\mathrm{Fun}(\mathfrak{g})^G$ (with small support), the following holds in $\hat{\mathcal{U}}(\mathfrak{g})$: (shhh, $\omega^2=j^{1/2}$)

$$\iint_{\mathfrak{g}\times\mathfrak{g}}\phi(x)\psi(y)\omega^2_{x+y}e^{x+y}=\iint_{\mathfrak{g}\times\mathfrak{g}}\phi(x)\psi(y)\omega^2_x\omega^2_y e^x e^y.$$

(shhh, this is Duflo)

Unitary statement. There exists $\omega\in\mathrm{Fun}(\mathfrak{g})^G$ and a (infinite order) tangential differential operator V defined on $\mathrm{Fun}(\mathfrak{g}_x\times\mathfrak{g}_y)$ so that

(1) $V\widehat{e^{x+y}}=\widehat{e^x}\widehat{e^y}V$ (allowing $\hat{\mathcal{U}}(\mathfrak{g})$-valued functions)

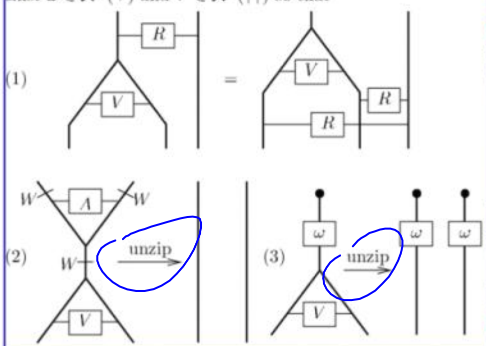
(2) $VV^*=I$ (3) $V\omega_{x+y}=\omega_x\omega_y$

Algebraic statement. With $I\mathfrak{g}:=\mathfrak{g}^*\rtimes\mathfrak{g}$, with $c:\hat{\mathcal{U}}(I\mathfrak{g})\to\hat{\mathcal{U}}(I\mathfrak{g})/\hat{\mathcal{U}}(\mathfrak{g})=\hat{\mathcal{S}}(\mathfrak{g}^*)$ the obvious projection, with S the antipode of $\hat{\mathcal{U}}(I\mathfrak{g})$, with W the automorphism of $\hat{\mathcal{U}}(I\mathfrak{g})$ induced by flipping the sign of $\mathfrak{g}^*$, with $r\in\mathfrak{g}^*\otimes\mathfrak{g}$ the identity element and with $R=e^r\in\hat{\mathcal{U}}(I\mathfrak{g})\otimes\hat{\mathcal{U}}(\mathfrak{g})$ there exist $\omega\in\hat{\mathcal{S}}(\mathfrak{g}^*)$ and $V\in\hat{\mathcal{U}}(I\mathfrak{g})^{\otimes 2}$ so that

(1) $V(\Delta\otimes 1)(R)=R^{13}R^{23}V$ in $\hat{\mathcal{U}}(I\mathfrak{g})^{\otimes 2}\otimes\hat{\mathcal{U}}(\mathfrak{g})$

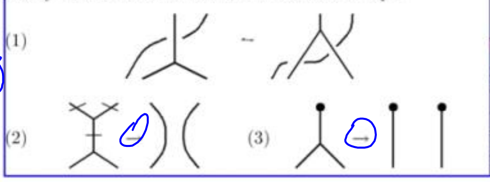
(2) $V\cdot SWV=1$ (3) $c(V\Delta(\omega))=\omega\otimes\omega$

Diagrammatic statement. Let $R=\exp\mathsf{H}\in\mathcal{A}^w(\uparrow\uparrow)$. There exist $\omega\in\mathcal{A}^w(\uparrow)$ and $V\in\mathcal{A}^w(\uparrow\uparrow)$ so that



turn green

Knot-Theoretic statement. There exists a homomorphic expansion Z for trivalent w-tangles. In particular, Z should satisfy $R4$ and intertwine annulus and disk unzips:



The Orbit Method. By Fourier analysis, the characters of $(\mathrm{Fun}(\mathfrak{g})^G,\star)$ correspond to coadjoint orbits in $\mathfrak{g}^*$. By averaging representation matrices and using Schur's lemma to replace intertwiners by scalars, to every irreducible representation of G we can assign a character of $(\mathrm{Fun}(G)^G,\star)$.

Subject flow chart

Convolutions statement ↔ The Orbit Method

Convolutions statement ↔ Group-Ring statement ↔ Unitary statement ↔ Algebraic statement ↔ Diagrammatic statement ↔ Knot-Theoretic statement

Convolutions statement → Free Lie statement ↔ Alekseev Torossian statement ↔ True

Diagrammatic statement ↔ Alekseev Torossian statement

Measure theoretic statement. Ignoring all ω's, there exists a measure preserving and orbit preserving transformation $T:\mathfrak{g}_x\times\mathfrak{g}_y\to\mathfrak{g}_x\times\mathfrak{g}_y$ for which $e^{x+y}\circ T=e^xe^y$.

Free Lie statement. There exist convergent Lie series F and G so that

$$x+y-\log e^ye^x=(1-e^{-\operatorname{ad}x})F+(e^{\operatorname{ad}y}-1)G$$

$$\operatorname{tr}(\operatorname{ad}x)\partial_xF+\operatorname{tr}(\operatorname{ad}y)\partial_yG=\frac{1}{2}\operatorname{tr}\left(\frac{\operatorname{ad}x}{e^{\operatorname{ad}x}-1}+\frac{\operatorname{ad}y}{e^{\operatorname{ad}y}-1}-\frac{\operatorname{ad}z}{e^{\operatorname{ad}z}-1}-1\right)$$

Alekseev-Torossian statement. There is an element $F\in\mathrm{TAut}_2$ with

$$F(x+y)=\log e^xe^y$$

and $j(F)\in\operatorname{im}\tilde\delta\subset\mathrm{tr}_2$, where for $a\in\mathrm{tr}_1$, $\tilde\delta(a):=a(x)+a(y)-a(\log e^xe^y)$.

Convolutions and Group Rings (ignoring all Jacobians). If G is finite, $(\mathrm{Fun}(G),\star)\cong(\mathbb{R}G,\cdot)$ via $T:f\mapsto\sum f(a)\tau(a)$. For Lie $\mathfrak{g}$ and G,

$$\begin{array}{ccc} (\mathfrak{g},+)\ni x & \xrightarrow{\tau} & e^x\in\hat{\mathcal{S}}(\mathfrak{g})\\ \downarrow{\scriptstyle\exp} & & \downarrow{\scriptstyle\chi}\\ (G,\cdot)\ni e^x & \xrightarrow{\tau} & e^x\in\hat{\mathcal{U}}(\mathfrak{g})\end{array}\quad\text{so}\quad\begin{array}{ccc}\psi\in\mathrm{Fun}(\mathfrak{g}) & \xrightarrow{T} & \hat{\mathcal{S}}(\mathfrak{g})\\ \downarrow{\scriptstyle\Phi^{-1}} & & \downarrow{\scriptstyle\chi}\\ \mathrm{Fun}(G) & \xrightarrow{T} & \hat{\mathcal{U}}(\mathfrak{g})\end{array}$$

with $T\psi=\int\psi(x)e^xdx\in\hat{\mathcal{S}}(\mathfrak{g})$ and $T\Phi^{-1}\psi=\int\psi(x)e^x\in\hat{\mathcal{U}}(\mathfrak{g})$. Given $\psi_i\in\mathrm{Fun}(\mathfrak{g})$ compare $\Phi^{-1}(\psi_1)\star\Phi^{-1}(\psi_2)$ and $\Phi^{-1}(\psi_1\star\psi_2)$ in $\hat{\mathcal{U}}(\mathfrak{g})$: (shhh, T is a "Fourier transform")

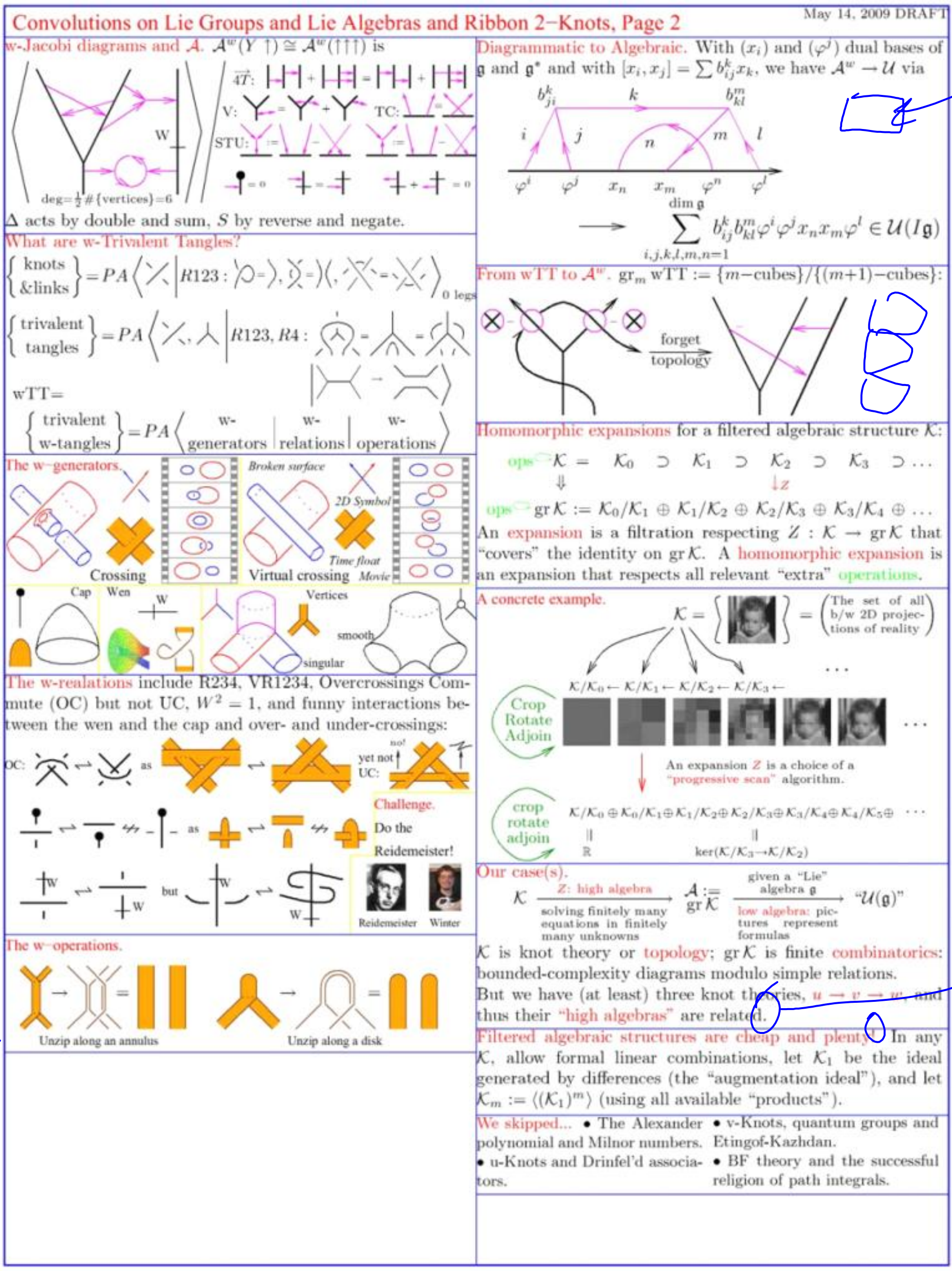
$$\star\text{ in }G:\iint\psi_1(x)\psi_2(y)e^xe^y\qquad\star\text{ in }\mathfrak{g}:\iint\psi_1(x)\psi_2(y)e^{x+y}$$

Unitary $\Longrightarrow$ Group-Ring. $\iint\omega^2_{x+y}e^{x+y}\phi(x)\psi(y)$

$$=\langle\omega_{x+y},\omega_{x+y}e^{x+y}\phi(x)\psi(y)\rangle=\langle V\omega_{x+y},Ve^{x+y}\phi(x)\psi(y)\omega_{x+y}\rangle$$
$$=\langle\omega_x\omega_y,e^xe^yV\phi(x)\psi(y)\omega_{x+y}\rangle=\langle\omega_x\omega_y,e^xe^y\phi(x)\psi(y)\omega_x\omega_y\rangle$$
$$=\iint\omega^2_x\omega^2_ye^xe^y\phi(x)\psi(y).$$

Unitary $\Longleftrightarrow$ Algebraic. The key is to interpret $\hat{\mathcal{U}}(I\mathfrak{g})$ as tangential differential operators on $\mathrm{Fun}(\mathfrak{g})$:

- $\varphi\in\mathfrak{g}^*$ becomes a multiplication operator.
- $x\in\mathfrak{g}$ becomes a tangential derivation, in the direction of the action of $\operatorname{ad}x$: $(x\varphi)(y):=\varphi([x,y])$.
- c is now "the constant term".



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w-Jacobi diagrams and $\mathcal{A}$. $\mathcal{A}^w(Y\uparrow) \cong \mathcal{A}^w(\uparrow\uparrow\uparrow)$ is

Δ acts by double and sum, S by reverse and negate.

What are w-Trivalent Tangles?

$\left\{\begin{matrix}\text{knots}\\ \text{\&links}\end{matrix}\right\} = PA\langle \ | R123 \rangle$ (0 legs)

$\left\{\begin{matrix}\text{trivalent}\\ \text{tangles}\end{matrix}\right\} = PA\langle \ | R123, R4 \rangle$

wTT= $\left\{\begin{matrix}\text{trivalent}\\ \text{w-tangles}\end{matrix}\right\} = PA\langle$ w-generators | w-relations | w-operations $\rangle$

The w-generators.

The w-realations include R234, VR1234, Overcrossings Commute (OC) but not UC, $W^2=1$, and funny interactions between the wen and the cap and over- and under-crossings:

The w-operations.

Diagrammatic to Algebraic. With (x_i) and (φ^j) dual bases of $\mathfrak{g}$ and $\mathfrak{g}^*$ and with $[x_i,x_j]=\sum b^k_{ij}x_k$, we have $\mathcal{A}^w \to \mathcal{U}$ via

$$\longrightarrow \sum_{i,j,k,l,m,n=1}^{\dim\mathfrak{g}} b^k_{ij}b^m_{kl}\varphi^i\varphi^j x_n x_m \varphi^l \in \mathcal{U}(I\mathfrak{g})$$

From wTT to $\mathcal{A}^w$. $\mathrm{gr}_m\,\mathrm{wTT} := \{m\text{–cubes}\}/\{(m+1)\text{–cubes}\}$:

Homomorphic expansions for a filtered algebraic structure $\mathcal{K}$:

$$\text{ops} \circlearrowright \mathcal{K} = \mathcal{K}_0 \supset \mathcal{K}_1 \supset \mathcal{K}_2 \supset \mathcal{K}_3 \supset \dots$$
$$\Downarrow \qquad \downarrow Z$$
$$\text{ops} \circlearrowright \mathrm{gr}\,\mathcal{K} := \mathcal{K}_0/\mathcal{K}_1 \oplus \mathcal{K}_1/\mathcal{K}_2 \oplus \mathcal{K}_2/\mathcal{K}_3 \oplus \mathcal{K}_3/\mathcal{K}_4 \oplus \dots$$

An expansion is a filtration respecting $Z:\mathcal{K}\to\mathrm{gr}\,\mathcal{K}$ that "covers" the identity on $\mathrm{gr}\,\mathcal{K}$. A homomorphic expansion is an expansion that respects all relevant "extra" operations.

A concrete example. $\mathcal{K} = \{\ \} = \left(\begin{matrix}\text{The set of all}\\ \text{b/w 2D projec-}\\ \text{tions of reality}\end{matrix}\right)$

$\mathcal{K}/\mathcal{K}_0 \leftarrow \mathcal{K}/\mathcal{K}_1 \leftarrow \mathcal{K}/\mathcal{K}_2 \leftarrow \mathcal{K}/\mathcal{K}_3 \leftarrow$ …

Crop Rotate Adjoin

An expansion Z is a choice of a "progressive scan" algorithm.

crop rotate adjoin $\quad \mathcal{K}/\mathcal{K}_0 \oplus \mathcal{K}_0/\mathcal{K}_1 \oplus \mathcal{K}_1/\mathcal{K}_2 \oplus \mathcal{K}_2/\mathcal{K}_3 \oplus \mathcal{K}_3/\mathcal{K}_4 \oplus \mathcal{K}_4/\mathcal{K}_5 \oplus \cdots$

$\| \quad \mathbb{R} \qquad \| \quad \ker(\mathcal{K}/\mathcal{K}_3 \to \mathcal{K}/\mathcal{K}_2)$

Our case(s). $\mathcal{K} \xrightarrow[\text{solving finitely many equations in finitely many unknowns}]{Z\text{: high algebra}} \mathcal{A} := \mathrm{gr}\,\mathcal{K} \xrightarrow[\text{low algebra: pictures represent formulas}]{\text{given a "Lie" algebra } \mathfrak{g}} \text{"}\mathcal{U}(\mathfrak{g})\text{"}$

$\mathcal{K}$ is knot theory or topology; $\mathrm{gr}\,\mathcal{K}$ is finite combinatorics: bounded-complexity diagrams modulo simple relations.

But we have (at least) three knot theories, $u \to v \to w$, and thus their "high algebras" are related.

Filtered algebraic structures are cheap and plenty. In any $\mathcal{K}$, allow formal linear combinations, let $\mathcal{K}_1$ be the ideal generated by differences (the "augmentation ideal"), and let $\mathcal{K}_m := \langle(\mathcal{K}_1)^m\rangle$ (using all available "products").

We skipped... • The Alexander polynomial and Milnor numbers. • u-Knots and Drinfel'd associators. • v-Knots, quantum groups and Etingof-Kazhdan. • BF theory and the successful religion of path integrals.

Penrose

Vass. Gous. Polyak

add the relation

make ops green

make "1" remove "1"

still missing: 1. Relation with A-T.

Fix Browsing order!