The Fourier Transform on Groups

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$$f \in \mathrm{Fun}(G) \longmapsto \int f(x) \underbrace{e^{x}}_{\text{in } \hat{\mathcal{U}}(\mathfrak{g})} dx \in \hat{\mathcal{U}}(\mathfrak{g})$$

Warmup: The Fourier transform on vector spaces:

$$f \in \mathrm{Fun}(V) \longmapsto \tilde{f} := \int f(x) \underbrace{e^{ix}}_{\text{in } \hat{S}(V)} dx \in \hat{S}(V)$$

Properties:

1. $f = \delta_{x_0} \Rightarrow \tilde{f} = e^{ix_0}$
2. If $v \in V$ and D_v is the directional derivative in the direction of v, then

$$\widetilde{D_v f} = -iv\tilde{f}$$

$\underline{\text{Pf}}$ $\widetilde{D_v f} = \int (D_v f) e^{ix} dx = -\int f D_v e^{ix} dx$

$= -i\left(\int f e^{ix} dx\right) v =$ as claimed.

$\underline{\text{Aside}}$ $D_v e^{ix} = \left.\frac{e^{i(x+\epsilon v)} - e^{ix}}{\epsilon}\right|_{\epsilon \to 0} = e^{ix} \left.\frac{e^{i\epsilon v} - 1}{\epsilon}\right|_{\epsilon \to 0}$

$= i e^{ix} v$