Glasgow Handout

March-27-09
9:17 AM

Convolutions on Lie Groups and Lie Algebras and Ribbon 2-Knots

Dror Bar-Natan, Glasgow April 2009, http://www.math.toronto.edu/~drorbn/Talks/Glasgow-0904

"God created the knots, all else in topology is the work of mortals." Leopold Kronecker (modified)

Convolutions statement. Convolutions of invariant functions on a Lie group agree with convolutions of invariant functions on its Lie algebra. More accurately, let G be a finite dimensional Lie group and let $\mathfrak{g}$ be its Lie algebra, let $j:\mathfrak{g}\to\mathbb{R}$ be the Jacobian of the exponential map $\exp:\mathfrak{g}\to G$, and let $\Phi:\mathrm{Fun}(G)\to\mathrm{Fun}(\mathfrak{g})$ be given by $\Phi(f)(x):=j^{1/2}(x)f(\exp x)$. Then if $f,g\in\mathrm{Fun}(G)$ are Ad-invariant and supported near the identity, then

$$\Phi(f)\star\Phi(g)=\Phi(f\star g).$$

Group-Ring statement. There exists $\omega\in\mathrm{Fun}(\mathfrak{g})^G$ so that for every $\phi,\psi\in\mathrm{Fun}(\mathfrak{g})^G$ (with small support), the following holds in $\mathcal{U}(\mathfrak{g})$:

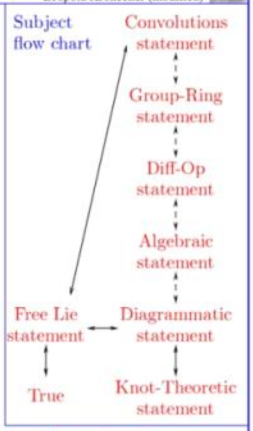
$$\iint_{\mathfrak{g}\times\mathfrak{g}}\phi(x)\psi(y)\omega(x+y)e^{x+y}=\iint_{\mathfrak{g}\times\mathfrak{g}}\phi(x)\psi(y)\omega(x)\omega(y)e^{x}e^{y}.$$

Diff op statement

There exists $\omega\in\mathrm{Fun}(g)^G$ and a tangential (infinite order?) diff. op V defined on $\mathrm{Fun}(g\times g)$, so that

$V_0=\omega(x+y)$ $\quad V_0^*=\omega(x)\omega(y)$

$\widehat{e^{x+y}}V=V\widehat{e^x}\widehat{e^y}$



Free Lie statement.

Diff-Op statement.

Algebraic statement.

Diagrammatic statement.

Knot-Theoretic statement.

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$$\Phi(f) \star \Phi(g) = \Phi(f \star g).$$

$\phi * \psi = \Phi(\Phi^{-1}(\phi) * \Phi^{-1}(\psi))$ $\phi, \psi \in \mathrm{Fun}(\mathfrak{g})$

$\Phi^{-1}(\phi * \psi) = \Phi^{-1}(\phi) * \Phi^{-1}(\psi)$ in $\mathrm{Fun}(G)$

bring flow chart back

would-be proof scheme:

Do; only then consider horizontals

$\mathrm{Fun}(\mathfrak{g})$	$\mathrm{Fun}(G)$	all verticals are algebra maps.
$\downarrow$	$\downarrow$	
$M(\mathfrak{g})$	$M(G)$	
$\downarrow$	$\downarrow$	
$S(\mathfrak{g})$	$U(\mathfrak{g})$	

Rephrase all in the unitary context

D

yet save a copy for recycling

Group-Ring statement. There exists $\omega \in \mathrm{Fun}(\mathfrak{g})^G$ so that for every $\phi, \psi \in \mathrm{Fun}(\mathfrak{g})^G$ (with small support), the following holds in $\mathcal{U}(\mathfrak{g})$: (shhh, $\omega = j^{1/2}$)

$$\iint_{\mathfrak{g}\times\mathfrak{g}} \phi(x)\psi(y)\omega(x+y)e^{x+y} = \iint_{\mathfrak{g}\times\mathfrak{g}} \phi(x)\psi(y)\omega(x)\omega(y)e^{x}e^{y}.$$

~~In the proud tradition of QFT, the art of avoiding path integration~~

Alekseev-Torossian statement.

Diff-Op statement. There exists $\omega \in \mathrm{Fun}(\mathfrak{g})^G$ and a tangential (infinite order) differential operator V defined on $\mathrm{Fun}(\mathfrak{g}_x \times \mathfrak{g}_y)$ so that $V_0 = \omega(x+y)$, $V_0^* = \omega(x)\omega(y)$ and so that when $\mathcal{U}(\mathfrak{g})$-valued functions are allowed,

$$\widehat{e^{x+y}}V = V\widehat{e^{x}}\widehat{e^{y}}.$$

V_0 is just $V1$

Free Lie statement. There exist convergent Lie series F and G so that

$$x + y - \log e^y e^x = (1 - e^{-\operatorname{ad} x})F + (e^{\operatorname{ad} y} - 1)G$$

$$\operatorname{tr}(\operatorname{ad} x)\partial_x F + \operatorname{tr}(\operatorname{ad} y)\partial_y G = \frac{1}{2}\operatorname{tr}\left(\frac{\operatorname{ad} x}{e^{\operatorname{ad} x} - 1} + \frac{\operatorname{ad} y}{e^{\operatorname{ad} y} - 1} - \frac{\operatorname{ad} z}{e^{\operatorname{ad} z} - 1} - 1\right)$$

Algebraic statement. With $I\mathfrak{g} := \mathfrak{g}^* \rtimes \mathfrak{g}$, with $i_+ : \mathcal{U}(I\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}^*) = \mathcal{S}(\mathfrak{g}^*)$ the obvious left $\mathcal{U}(I\mathfrak{g})$-module morphism, with S the antipode of $\mathcal{U}(I\mathfrak{g})$, with W the automorphism of $\mathcal{U}(I\mathfrak{g})$ induced by flipping the sign of $\mathfrak{g}^*$, with $r \in \mathfrak{g}^* \otimes \mathfrak{g}$ the identity element and with $R = e^r \in \mathcal{U}(I\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ there exist $\omega \in \mathcal{S}(\mathfrak{g}^*)$ and $V \in \mathcal{U}(I\mathfrak{g})^{\otimes 2}$ so that $i_+ V = \Delta(\omega)$, $i_+ SWV = \omega \otimes \omega$ and in $\mathcal{U}(I\mathfrak{g})^{\otimes 2} \otimes \mathcal{U}(\mathfrak{g})$,

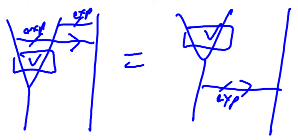
$$(\Delta \otimes 1)(R)V = VR^{13}R^{23}.$$

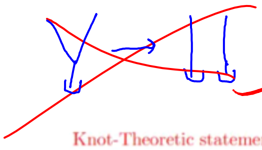
y

$$\int \widehat{e^{x+y}}\, V \phi\psi = \int V \widehat{e^{x}}\widehat{e^{y}} \phi\psi$$

$$\int e^{x+y} w(x+y) \phi\psi = \int w(x) w(y) e^{x} e^{y} \phi\psi$$

Diagrammatic statement.





Knot-Theoretic statement.

* Arrange a mechanism for a third page "drafts/recycling".

* Inset of the vvw handout? (No space)

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Group-Ring statement. There exists $\omega\in\mathrm{Fun}(\mathfrak{g})^G$ so that for every $\phi,\psi\in\mathrm{Fun}(\mathfrak{g})^G$ (with small support), the following holds in $\mathcal{U}(\mathfrak{g})$: (shhh, $\omega=j^{1/2}$)

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Diff-Op statement. There exists $\omega\in\mathrm{Fun}(\mathfrak{g})^G$ and a tangential (infinite order) unitary ($V^{-1}=V^*$) differential operator V defined on $\mathrm{Fun}(\mathfrak{g}_x\times\mathfrak{g}_y)$ so that $V\omega(x+y)=\omega(x)\omega(y)$ and so that when $\mathcal{U}(\mathfrak{g})$-valued functions are allowed,

add $V1=1$

$$V\widehat{e^{x+y}}=\widehat{e^x}\widehat{e^y}V.$$

Algebraic statement. With $I\mathfrak{g}:=\mathfrak{g}^*\rtimes\mathfrak{g}$, with $i_+:\mathcal{U}(I\mathfrak{g})\to\mathcal{U}(\mathfrak{g}^*)=\mathcal{S}(\mathfrak{g}^*)$ the obvious left $\mathcal{U}(I\mathfrak{g})$-module morphism, with S the antipode of $\mathcal{U}(I\mathfrak{g})$, with W the automorphism of $\mathcal{U}(I\mathfrak{g})$ induced by flipping the sign of $\mathfrak{g}^*$, with $r\in\mathfrak{g}^*\otimes\mathfrak{g}$ the identity element and with $R=e^r\in\mathcal{U}(I\mathfrak{g})\otimes\mathcal{U}(\mathfrak{g})$ there exist $\omega\in\mathcal{S}(\mathfrak{g}^*)$ and $V\in\mathcal{U}(I\mathfrak{g})^{\otimes 2}$ so that $V^{-1}=V^*:=SWV$, $i_+(V)=1$, and $i_+(V\Delta(\omega))=\omega\otimes\omega$, and in $\mathcal{U}(I\mathfrak{g})^{\otimes 2}\otimes\mathcal{U}(\mathfrak{g})$,

$$V(\Delta\otimes 1)(R)=R^{13}R^{23}V.$$

Diagrammatic statement.

Knot-Theoretic statement.

The orbit method

Subject flow chart

"The orbit method"

Convolutions statement ↔ Group-Ring statement ↔ Diff-Op statement ↔ Algebraic statement ↔ Diagrammatic statement ↔ Knot-Theoretic statement

Convolutions statement → Free Lie statement

Free Lie statement ↔ Alekseev Torossian statement ↔ True

Alekseev Torossian statement → Diagrammatic statement

Free Lie statement. There exist convergent Lie series F and G so that

$$x+y-\log e^ye^x=(1-e^{-\operatorname{ad}x})F+(e^{\operatorname{ad}y}-1)G$$

$$\operatorname{tr}(\operatorname{ad}x)\partial_xF+\operatorname{tr}(\operatorname{ad}y)\partial_yG=\frac{1}{2}\operatorname{tr}\left(\frac{\operatorname{ad}x}{e^{\operatorname{ad}x}-1}+\frac{\operatorname{ad}y}{e^{\operatorname{ad}y}-1}-\frac{\operatorname{ad}z}{e^{\operatorname{ad}z}-1}-1\right)$$

Alekseev-Torossian statement.

Add a formal ($\hbar$ or t) parameter?

1. In $e^{\hbar x}$?
2. In $xy-yx=t[x,y]$?

not for now

Diff-Op $\Longrightarrow$ Group-Ring.

$$\int \omega(x+y)e^{x+y}\phi(x)\psi(y) = \left\langle \omega(x+y), \widehat{e^{x+y}}\phi(x)\psi(y) \right\rangle$$
$$= \left\langle V\omega(x+y), V\widehat{e^{x+y}}\phi(x)\psi(y) \right\rangle = \langle \omega(x)\omega(y), \widehat{e^x}\widehat{e^y}V\phi(x)\psi(y)\rangle$$
$$= \langle \omega(x)\omega(y), \widehat{e^x}\widehat{e^y}\phi(x)\psi(y)\rangle = \int \omega(x)\omega(y)e^x e^y\phi(x)\psi(y)$$

damn; especially,
$V w(x+y) = w(x)w(y)$
& $V w(x)w(y) = w(x)w(y)$
don't go together well.
[or do They?]

$j^{+1/2}$

dashed

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$$\iint_{\mathfrak{g}\times\mathfrak{g}}\phi(x)\psi(y)\omega^2(x+y)e^{x+y}=\iint_{\mathfrak{g}\times\mathfrak{g}}\phi(x)\psi(y)\omega^2(x)\omega^2(y)e^xe^y.$$

Diff-Op statement. There exists $\omega\in\mathrm{Fun}(\mathfrak{g})^G$ and a (infinite order) unitary ($V^{-1}=V^*$) tangential differential operator V defined on $\mathrm{Fun}(\mathfrak{g}_x\times\mathfrak{g}_y)$ so that $V\omega(x+y)=\omega(x)\omega(y)$ and so that when $\mathcal{U}(\mathfrak{g})$-valued functions are allowed,

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Algebraic statement. With $I\mathfrak{g}:=\mathfrak{g}^*\rtimes\mathfrak{g}$, with $c:\mathcal{U}(I\mathfrak{g})\to\mathcal{U}(I\mathfrak{g})/\mathcal{U}(\mathfrak{g})=\mathcal{S}(\mathfrak{g}^*)$ the obvious projection, with S the antipode of $\mathcal{U}(I\mathfrak{g})$, with W the automorphism of $\mathcal{U}(I\mathfrak{g})$ induced by flipping the sign of $\mathfrak{g}^*$, with $r\in\mathfrak{g}^*\otimes\mathfrak{g}$ the identity element and with $R=e^r\in\mathcal{U}(I\mathfrak{g})\otimes\mathcal{U}(\mathfrak{g})$ there exist $\omega\in\mathcal{S}(\mathfrak{g}^*)$ and $V\in\mathcal{U}(I\mathfrak{g})^{\otimes 2}$ so that $V^{-1}=V^*:=SWV$, $c(V\Delta(\omega))=\omega\otimes\omega$, and in $\mathcal{U}(I\mathfrak{g})^{\otimes 2}\otimes\mathcal{U}(\mathfrak{g})$,

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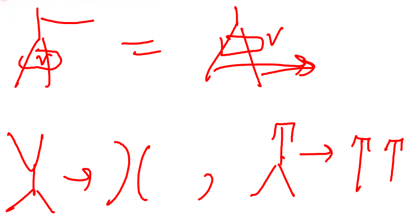
Diagrammatic statement.



Knot-Theoretic statement. There exists a homomorphic expansion Z for w-tangled trivalent graphs.

A full description of w-knots should come here

The Orbit Method. By Fourier analysis, the charcters of $(\mathrm{Fun}(\mathfrak{g})^G,\star)$ correspond to coadjoint orbits in $\mathfrak{g}^*$. By averaging representation matrices and using Schur's lemma to replace intertwiners by scalars, to every irreducible representation of G we can assign a character of $(\mathrm{Fun}(G)^G,\star)$.

$\boxed{K}$ $\boxed{V}$

Subject flow chart

- Convolutions statement ↔ The Orbit Method
- Convolutions statement ↔ Free Lie statement
- Convolutions statement ↔ Group-Ring statement ↔ Diff-Op statement ↔ Algebraic statement ↔ Diagrammatic statement ↔ Knot-Theoretic statement
- Free Lie statement ↔ Alekseev Torossian statement ↔ True
- Diagrammatic statement ↔ Alekseev Torossian statement

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Alekseev-Torossian statement.

$\exists F\in TAut_2$ with $F(x+y)=\log e^xe^y$ and $j(F)\in\operatorname{im}\tilde\delta$

where $\tilde{F}(a(x)+a(y))-a(\log e^xe^y)$

$\boxed{A}$ $\boxed{T}$

The dictionary with ribbon 2-knots should come here.

remove IAS seal

add: "tangential ⇒ commutes with invariants" and hence

Convolutions on Lie Groups and Lie Algebras and Ribbon 2–Knots, Page 2

Diff-Op $\Longrightarrow$ Group-Ring.

$$\begin{aligned}\int\omega^2(x+y)e^{x+y}\phi(x)\psi(y)&=\left\langle\omega(x+y),\omega(x+y)\widehat{e^{x+y}}\phi(x)\psi(y)\right\rangle\\&=\left\langle V\omega(x+y),V\omega(x+y)\widehat{e^{x+y}}\phi(x)\psi(y)\right\rangle\\&=\langle\omega(x)\omega(y),\widehat{e^x}\widehat{e^y}V\omega(x+y)\phi(x)\psi(y)\rangle\\&=\langle\omega(x)\omega(y),\widehat{e^x}\widehat{e^y}\omega(x)\omega(y)\phi(x)\psi(y)\rangle\\&=\int\omega^2(x)\omega^2(y)e^xe^y\phi(x)\psi(y).\end{aligned}$$

compress

Further boxes:
*convolutions and group ring

add: "tangential ⇒ commutes with invariants" and hence

Diff-Op $\Longrightarrow$ Group-Ring. $\int \omega^2(x+y)e^{x+y}\phi(x)\psi(y)$

$$
\begin{aligned}
&= \left\langle \omega(x+y), \omega(x+y)\widehat{e^{x+y}}\phi(x)\psi(y) \right\rangle \\
&= \left\langle V\omega(x+y), V\omega(x+y)\widehat{e^{x+y}}\phi(x)\psi(y) \right\rangle \\
&= \langle \omega(x)\omega(y), \widehat{e^x}\widehat{e^y}V\omega(x+y)\phi(x)\psi(y) \rangle \\
&= \langle \omega(x)\omega(y), \widehat{e^x}\widehat{e^y}\omega(x)\omega(y)\phi(x)\psi(y) \rangle \\
&= \int \omega^2(x)\omega^2(y)e^x e^y\phi(x)\psi(y).
\end{aligned}
$$

compress

Further boxes;

* convolutions and group ring
* Diff op and Algebraic
* Algebraic and Diagrammatic
* Grrr
* Homomorphic Expansions
* Diagrammatic and A-T

Draft

* Figure out how to do equation spacing right in LaTeX. ✓