Categorical Free Group Theory

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Challenge Prove in a categorical way that if $a^{-1}xa = x$, then $a = x^n$.

$$\begin{tikzcd} \mathbb{Z}[x] \arrow[r, hook, "i"] & \mathbb{Z}F \arrow[l, bend right, "\ell_x"'] \arrow[r, "c"] & \mathbb{Z}F \arrow[l, bend right, "\partial"'] \end{tikzcd}$$

$$a \longmapsto a^{-1}xa$$

i.e., $a \mapsto m\circ(1\otimes L_x)\circ(S\otimes 1)\circ\Delta(a)$

i is the inclusion $\quad$ c is "conjugation"

ℓ_x forgets all letters except x.

∂w maps w to the sum of all ways of cutting it at an x and taking the remaining tail. $\Rightarrow$ Letter take "sum of right cuts + sum of inverses of left cuts"

<u>Claim</u>

$$(\ell_x\circ i) = I$$

$$i\circ\ell_x + \partial\circ c = \text{not working.}$$

Second attempt: Now take $\partial w = \frac{1}{2}(\partial_L w + S(\partial_R w))$ where:

possibly left or right multiplied by $x^{\pm1}$

$\partial_L w$ = sum of all ways of cutting w at an x and taking the remaining tail. $\quad (\partial_L xx^{-1}w) = x^{-1}w + (\partial_L x^{-1})w + \partial_L w$

$\Rightarrow \partial_L x^{-1} = -x^{-1}$

$\partial_R w$ = sum of all ways of cutting w at an x and taking the remaining head.

<u>Claim</u> $\quad \partial\circ c + i\circ\ell_x =$

$$r^{-1}i^{-1}x^{-1}i^{-1}n^{-1}m^{-1}a^{-1}k^{-1}xkamnixer \longmapsto er + kamnixer - x^{-1}i^{-1}n^{-1}\ldots$$

$$+x^{-1}S\left(-r^{-1}e^{-1}x^{-1}+(\text{kamnixer})^{-1}+(\text{kamnixer})^{-1}x\,\text{kam}_{ni}\right)$$

cancellation error

$$\neq (1+x^{-1})\,\text{kamnixer}$$

$$\partial_L(a\cdot b)=\partial_L(a)\cdot b+\partial_L(b) \qquad \partial_L(x)=1$$

$$\partial_L(x^{-1})=-x^{-1}$$

$$\partial_L(xx^{-1})=x^{-1}-x^{-1}=0$$

$$\partial_L(x^{-1}x)=-x^{-1}x+1=0$$

$$\partial_R(ab)=\partial_R(a)+a\partial_R(b) \qquad \partial_R(x)=1$$

$$\partial_R(x^{-1})=-x^{-1}$$

$$\partial_R(xx^{-1})=-xx^{-1}+1=0$$

$$\partial_R(x^{-1}x)=-x^{-1}+x^{-1}=0$$

$$\partial=\left(\partial_L+L_{x^{-1}}\cdot S(\partial_R)\right)/2$$

$$2(\partial\cdot c)(axb)=2\partial(b^{-1}x^{-1}a^{-1}xaxb)$$

$$=-\cancel{x^{-1}a^{-1}xa}xb+axb\cancel{+b}$$

$$+x^{-1}S\left(-\cancel{b^{-1}x^{-1}}+b^{-1}x^{-1}a^{-1}+\cancel{b^{-1}x^{-1}a^{-1}xa}\right)$$

no cancellation.

$$=(1+x^{-1})axb$$

$$2(\partial\cdot c)(x^{7})=2\partial(x)=1+x^{-1}$$

$$2(\partial\cdot c)(x^{7}a)=2\partial(a^{-1}xa)=a+x^{-1}a=(1+x^{-1})a$$

$$2\partial(x^{-1}xx)=-x^{-1}xx+x+1$$

$$+x^{-1}S(-x^{-1}+x^{-1}+x^{-1}x)$$

$$=-\cancel{x^{-1}x}x\cancel{+x}+1-\cancel{x^{-1}x}\cancel{+x^{-1}}x+x^{-1}$$

$$=1-x^{-1}$$

$$2\partial(\cancel{x^{-1}}\cancel{x^{-1}}\cancel{x^{-1}}\cancel{x^{-1}}x\cancel{x}\cancel{x}\cancel{x}\cancel{x})\neq x^{4}+x^{-1}x^{4}$$

x^{-1} ... $x^{-1}(x^{-1}x^{-1})^{-1}$ ... x ... $(x^{-1}x^{-1})^{-1}$

$-x^{-1}x^3 \qquad -x^{-1}\cdot(x^{-1}\cdot x^{-1})^{-1} \qquad x \qquad (x^{-1}x^{-1})^{-1}$

$-x^2 \qquad -x \qquad x + x^2$

$$\partial_L(x^{-1}xx) = -x^{-1}xx + x + 1 = 1$$

$$\partial_L(x) = 1 \qquad \partial_R(x) = 1$$

$$\partial_R(x^{-1}xx) = -x^{-1} + x^{-1} + x^{-1}x = 1$$

$$x^{-1}S(\partial_R(x^{-1}xx)) = -x^{-1}x + x^{-1}x + x^{-1}x^{-1}x$$