The Euler Parametrization

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11:08 AM

For a graded Lie algebra $\mathfrak{g}$, let $J(\phi)$ denote

$$J(\phi) = \frac{1-e^{-\mathrm{ad}\,\phi}}{\mathrm{ad}\,\phi}$$

and let E be the Euler operator, defined for homogeneous ψ by $E\psi = (\deg\psi)\psi$

We have:

$$\left\{ \begin{matrix}\text{standard parametrization} \\ x\in\mathfrak{g}\end{matrix} \right\} \rightleftarrows \left\{ \begin{matrix}\text{Euler parametrization} \\ \phi\in\mathfrak{g}\end{matrix} \right\}$$

by

$$J(\mathrm{ad}\,x)(Ex) = \phi$$

Given a $\overbrace{\text{Lie}}^{\text{graded}}$ algebra $\mathfrak{g}$ and $\phi, \psi \in \mathfrak{g}$ define

$$\phi \# \psi := e^{-\mathrm{ad}\,y}(\phi) + \psi$$

$$J(\mathrm{ad}\,x)(Ex) = \phi \qquad J(\mathrm{ad}\,y)(Ey) = \psi$$

$$\phi = e^{-x}Ee^{x} \qquad \psi = e^{-y}Ee^{y}$$

$$J(\mathrm{ad}\log e^{x}e^{y})(E\log e^{x}e^{y}) = e^{-y}e^{-x}E(e^{x}e^{y})$$

$$= e^{-y}e^{-x}E(e^{x})e^{y} + e^{-y}e^{-x}e^{x}E(e^{y})$$

$$= e^{-y}\phi e^{y} + \psi = J(\mathrm{ad}\,y)(\phi) + \psi$$

$$= e^{-\mathrm{ad}\,y}\phi + \psi$$

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tentative

$$\frac{e^{\mathrm{ad}\,y}-1}{\mathrm{ad}\,y}(Ey) = \psi$$

$$(e^{\mathrm{ad}\,y}-1)Ey = [y,\psi]$$

$$e^{\mathrm{ad}\,y}(Ey) - Ey = [y,\psi] \qquad \text{useless}$$

Inverting $J(\mathrm{ad}\,x)(Ex) = \phi$:

$$\phi = \phi_1 + \phi_2 + \dots \qquad x = x_1 + x_2 + \dots$$

$$J(y) = \frac{1-e^{-y}}{y} = 1 - \frac{y}{2} + \frac{y^2}{6} - \frac{y^3}{24} \dots$$

To degree 1:

$$x_1 = \phi_1$$

To degree 2:

$$2x_2 + 0 = \phi_2 \implies x_2 = \phi_2/2$$

To degree 3:

$$3x_3 - \frac{\mathrm{ad}\,x_1}{2}(2x_2) - \frac{\mathrm{ad}\,x_2}{2}x_1 = \phi_3$$

$$3x_3 - \frac{1}{2}[x_1, x_2] = \phi_3 \implies$$

$$x_3 = \frac{\phi_3}{3} + \frac{1}{12}[\phi_1, \phi_2]$$

To degree 4:

$$J(y) = \frac{1-e^{-y}}{y} = 1 + (J(y) - 1) = 1 + \left(\frac{1-e^{-y}}{y} - 1\right)$$
$$= 1 + \frac{1-y-e^{-y}}{y}$$