BCH modulo [[L,L],[L,L]]

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10:41 AM

Question Find a function f of x and y so that in $U\left(L/[[L,L],[L,L]]\right)$ $(L:=FL(x,y))$,

$$e^x e^y = \exp\left(x+y+f\dashv[x,y]\right)$$

the Euler operator

Solution Apply F, where $F(Z) := Z^{-1}EZ$:

As of Dec, 2008, this is G, the "flow".

$$F(e^xe^y) = e^{-y}e^{-x}E(e^xe^y) =$$
$$e^{-y}e^{-x}E(e^x)e^y + e^{-y}e^{-x}e^xE(e^y)$$
$$= e^{-y}xe^y + y$$
$$= e^{-\operatorname{ad} y}x + y$$
$$= \left(1+(e^{-\operatorname{ad} y}-1)\right)x + y$$
$$= x+y+\frac{1-e^{-y}}{y}\dashv[x,y]$$

Aside: E is a derivation: $E(fg) = E(f)g + fE(g)$

Aside: $E(e^x) = xe^x$

Aside: $e^{-B}Ae^B = e^{-\operatorname{ad} B}A$

scratch:
$$\frac{1}{x}\left(\frac{1-e^{-x}}{x}-1\right) = \frac{1}{x}\,\frac{1-x-e^{-x}}{x}$$
$$= \frac{1-x-e^{-x}}{x^2} =: J_2(x)$$
while $\frac{1-e^{-x}}{x} =: J(x)$

On the other hand,

$$F\left(\exp(x+y+f\dashv[x,y])\right)$$
$$= J\left(\operatorname{ad}(x+y+f\dashv[x,y])\right)\cdot\left(x+y+(2f+Ef)\dashv[x,y]\right)$$
$$= J_2\left(\operatorname{ad}(x+y+f\dashv[x,y])\right)\circ \operatorname{ad}(x+y+f\dashv[x,y])\left(x+y+(2+E)f\dashv[x,y]\right) + x+y+(2+E)f\dashv[x,y]$$
$$= J_2(-)\Big(\cancel{[x,y]} + x(2+E)f\dashv[x,y] + \cancel{[y,x]} + y(2+E)f\dashv[x,y] - xf\dashv[x,y] - yf\dashv[x,y]\Big) + x+y+(2+E)f\dashv[x,y]$$
$$= J_2(-)\left((x+y)(1+E)f\dashv[x,y]\right) + x+y+(2+E)f\dashv[x,y]$$

Aside:
$$\frac{d}{dt}e^{A(t)} = e^{A(t)}\,\frac{1-e^{-\operatorname{ad} A(t)}}{\operatorname{ad} A(t)}\,\frac{d}{dt}A(t)$$
and therefore
$$F(e^g) = \frac{1-e^{-\operatorname{ad} g}}{\operatorname{ad} g}Eg = J(\operatorname{ad} g)Eg$$

up to signs asides:

$$J(\mathrm{ad}(x+y+f\dashv[x,y]))(g\dashv[x,y]) = J(x+y)\,g\dashv[x,y]$$

$$J(\mathrm{ad}(x+y+f\dashv[x,y]))(x) = \frac{J(x+y)-J(x)}{y}\dashv[x,y]$$

$$+\, J'(x+y)\cdot f\cdot x\dashv[x,y]$$