

## Det and Tr

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3:37 PM

$$\chi_\lambda(A) := \det(\lambda I - A)$$

$$\begin{aligned}\text{Tr}(\lambda I - A)^{-1} &= \sum \frac{1}{\lambda - \lambda_i} \\ &= \sum \frac{d}{d\lambda} \log(\lambda - \lambda_i) \\ &= \frac{d}{d\lambda} \log \prod (\lambda - \lambda_i) \\ &= \frac{d}{d\lambda} \log \chi_\lambda(A) = \frac{d}{d\lambda} \log \det(\lambda I - A)\end{aligned}$$

Alternatively,

$$\begin{aligned}\frac{d}{d\lambda} \log(\det(\lambda I - A)) &= \frac{\frac{d}{d\lambda} \det(\lambda I - A)}{\det(\lambda I - A)} = \\ &= \frac{\det(\lambda I - A) \text{tr}(\lambda I - A)^{-1} I}{\det(\lambda I - A)} = \text{tr}(\lambda I - A)^{-1}\end{aligned}$$

In general,  $\frac{d}{d\epsilon} \text{tr} A_\epsilon = \text{tr} \frac{d}{d\epsilon} A_\epsilon$

$$\begin{aligned}\frac{d}{d\epsilon} \det A_\epsilon &= \frac{d}{d\epsilon} \det A_0 \det A_0^{-1} A_\epsilon \\ &= \det A_0 \text{tr} A_0^{-1} \frac{d}{d\epsilon} A_\epsilon\end{aligned}$$

Input:

$$B_{ij} = \begin{cases} X^{-s_j d_j} - 1 & \text{if head}(a_j) \\ & \text{is within the open span} \\ & \text{of } a_i \\ 0 & \text{otherwise} \end{cases}$$

log Gauss diagram by arrows  $a_1, a_n$  signs  $s_1, s_n$  & dir  $d_1, \dots, d_n$   $d_1 \rightarrow \rightarrow$   $d_n \rightarrow \rightarrow$

$X$  is a variable.

In our example,

$$B = (B_{ij}) = \begin{pmatrix} 0 & 0 & 0 \\ X^{-1} & 1 & 0 & X^{-1} \\ 0 & X^{-1} & 1 & 0 \end{pmatrix}; C = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Then  $Z = \text{Tr}((I-B)^{-1} B \cdot C)$  is an invariant of  $w$ -knots con. If  $A(x)$  is the Alex,

then  $Z = -X \frac{A'(X)}{A(X)} = -X (\log A(X))'$

$C = I + \frac{\partial}{\partial X} B|_{X=1}$   
 Verified on all knots up to 11 crossings inclusive.

**Warning.** The first term in the definition of  $C$ , written on the blackboard as the identity matrix  $I$ , should really be the diagonal matrix whose  $jj$  entry is  $-s_j d_j$ .

Thus the  $C$  in the example should be

$$\begin{pmatrix} -1 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$Z = -X \frac{A'}{A} = -X (\log A)' \Leftrightarrow \int \frac{Z}{X} dx = \log A$$

$$\Leftrightarrow A = \exp\left(-\int \frac{Z}{X} dx\right)$$