PROJECTIVIZATION, WELDED KNOTS AND ALEKSEEV-TOROSSIAN

[Summary of Oberwolfach talk, given May 9, 2008]

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Abstract. My talk had two parts:

- In the first part I described the (tentative and speculative) "Projectivization Paradigm", which says, roughly speaking, that everything graded and interesting is the associated graded of something plain ("ungraded", "global") and even more interesting. The paradigm is absolutely general, encompassing practically every algebraic structure that might exist, and there is a diverse base of interesting examples and candidates for future examples.
- In the second part I described my latest example of an instance of the Projectivization Paradigm: I showed that the projectivization of "the circuit algebra of welded tangles" describes a good part (and maybe, in the future, all) of the recent work by Alekseev and Torossian on Drinfel'd's associators and the Kashiwara-Vergne conjecture. This is cool: it leads to a nice conceptual construction of tree-level associators which might even be brought to a closed form, and it seems like a step towards a better understanding of quantum universal enveloping algebras and the work of Etingof and Kazhdan.

To a very large extent my talk followed the two-page handout attached as the last two pages of this document.

1. The Projectivization Speculative Paradigm

I started by reminding the conference about the "Categorification Speculative Paradigm", which says, in very rough terms, that all of mathematics, or at least all of integer-coefficient mathematics, is the "Euler shadow" of vector-space, homological, mathematics. This, of course, is merely a speculative paradigm — one cannot expect it to be literally true, yet it is an excellent guiding principle for research. A lot of interesting mathematics arises as one tries to explore the extent to which this speculative paradigm holds true.

In a similar manner I proposed the "Projectivization Tentative[1] Speculative Paradigm", ~~which says,~~ which says, in very rough terms, that all of graded mathematics is the projectivization of "plain", "ungraded"~~,~~ or "global" mathematics: all graded algebraic structures are the projectivizations of global ones, and all graded equations are the equations for "homomorphic expansions", or for "automorphisms" of homomorphic expansions.

I then proceeded to explain most of the terms appearing in the above paragraph. For a start, I gave a few examples of "graded equations" (these are the ~~things~~ entities the projectivization paradigm is supposed to explain):

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[1] "Tentative" because I'm not even sure if the name "projectivization" (meant to be catchy and convey a "graded" feeling) is appropriate.

- The exponential equation $e(x+y) = e(x)e(y)$ [BN4].
- The pentagon and hexagon equations [Dr2, Dr3, BN1, BN2] space. for Drinfeld associators.
- The equations defining a quantized universal enveloping algebra in the sense of Drinfel'd [Dr1] and Etingof-Kazhdan [EK]. For the long term, these are the equations I care about the most, and my dream is to eventually incorporate them to within the projectivization paradigm.
- The equations appearing in the Alekseev-Torossian work [AT] on Drinfeld associators and the Kashiwara-Vergne Conjecture [KV]. These equations are the main concern of the second part of this talk. One wonderful feature of these equations is that (in suitable quotients) they have explicit solutions, that will likely lead to explicit formulas for tree-level associators.

I then moved on to explain what is "the projectivization of an algebraic structure". For this purpose, an "algebraic structure" $\mathcal{O}$ is practically anything that is made of "spaces" and "operations". Allowing for formal linear combinations and extending all operations in a multi-linear manner, we can always define an "augmentation ideal" I $\mathcal{O}$ along with its powers I^n, and then we can set

$$\operatorname{proj}\mathcal{O} := \bigoplus_{n\geq 0} I^n/I^{n+1}.$$

One can see that $\operatorname{proj}\mathcal{O}$ is endowed with the same operations as $\mathcal{O}$, though they need not satisfy the same "axioms" that the operations of $\mathcal{O}$ may satisfy. We noted that if $\mathcal{O}$ is an appropriate space of knotted objects, then $\operatorname{proj}\mathcal{O}$ is the corresponding space of "chord diagrams".

Some warmup examples followed. We noted that the projectivization of a group is a graded associative algebra, and that the projectivization of a quandle is a graded Lie algebra.

I then moved on to discuss the central notion in the statement of the projectivization paradigm — the notion of an "expansion" [Li], and more importantly, of a "homomorphic expansion" — a "homomorphism" $Z\colon \mathcal{O} \to \operatorname{proj}\mathcal{O}$ which "covers" the identity map on $\operatorname{proj}\mathcal{O}$. When $\mathcal{O}$ is "finitely presented", finding an expansion involves finding values for $Z(g_i)$ (where the g_i's are the generators of $\mathcal{O}$), where these values must satisfy the equations corresponding to the "defining relations" of $\mathcal{O}$. Hence as promised[2] in the statement of the projectivization paradigm, finding a homomorphic expansion is a matter of solving equations in a graded space, $\operatorname{proj}\mathcal{O}$.

I then explained how homomorphic expansions may be used — they convert certain kinds of "global" problems into problems that can be addressed "degree by degree". A pretty example is "Algebraic Knot Theory" [BN3]. $\mathcal{O}$: Here the algebraic structure, ...

References

[AT] A. Alekseev and C. Torossian, *The Kashiwara-Vergne conjecture and Drinfeld's associators,*, arXiv:0802.4300.

[BN1] D. Bar-Natan, *Non-Associative Tangles,* in *Geometric topology* (proceedings of the Georgia international topology conference), (W. H. Kazez, ed.), 139–183, Amer. Math. Soc. and International Press, Providence, 1997.

[BN2] D. Bar-Natan, *On Associators and the Grothendieck-Teichmuller Group I,* Selecta Mathematica, New Series **4** (1998) 183–212.

[2]The other source of graded equations, "automorphisms of homomorphic expansions", was not discussed in my talk.

[BN3] D. Bar-Natan, *Algebraic Knot Theory — A Call for Action,* web document (2006), `http://www.math.toronto.edu/~drorbn/papers/AKT-CFA.html`.

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[Dr1] V. G. Drinfel'd, *Quantum Groups,* in *Proceedings of the International Congress of Mathematicians,* 798–820, Berkeley, 1986.

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Dror Bar-Natan: Talks: Oberwolfach-0805:

Projectivization, Welded Knots and Alekseev–Torossian

The Categorification Speculative Paradigm.

- Every object in math is the Euler characteristic of a complex.
- Every operation in math lifts to an operation between complexes.
- Every identity in mathematics is true up to homotopy.

The Projectivization Tentative Speculative Paradigm. Projectivization?

- Every graded algebraic structure in mathematics is the projectivization of a plain ("global") one.
- Every equation written in a graded algebraic structure is an equation for a homomorphic expansion, or for an automorphism of such.

Graded Equations Examples

- $e(x+y)=e(x)e(y)$ in $\mathbb{Q}[[x,y]]$.
- The pentagon and hexagons in $\mathcal{A}(\uparrow_{3,4})$.
- The equations defining a QUEA, the work of Etingof and Kazhdan.
- The Alekseev-Torossian equations in $\mathcal{U}(\mathrm{sder}_n)$ and $\mathcal{U}(\mathrm{tder}_n)$.

sder $\leftrightarrow$ tree-level $\mathcal{A}$
tder $\leftrightarrow$ more

$$F\in\mathcal{U}(\mathrm{tder}_2);\quad F^{-1}e(x+y)F=e(x)e(y)\quad\Longleftrightarrow\quad F\in\mathrm{Sol}_0$$

$$\Phi=\Phi_F:=(F^{12,3})^{-1}(F^{1,2})^{-1}F^{23}F^{1,23}\in\mathcal{U}(\mathrm{sder}_3)$$

$$\Phi^{1,2,3}\Phi^{1,23,4}\Phi^{2,3,4}=\Phi^{12,3,4}\Phi^{1,2,34}\qquad\text{“the pentagon”}$$

$$t=\frac{1}{2}(y,x)\in\mathrm{sder}_2\text{ satisfies }4T\quad\text{and}\quad r=(y,0)\in\mathrm{tder}_2\text{ satisfies }6T$$

$$R:=e(r)\text{ satisfies Yang-Baxter: }R^{12}R^{13}R^{23}=R^{23}R^{13}R^{12}$$

$$\text{also }R^{12,3}=R^{13}R^{23}\text{ and }F^{23}R^{1,23}(F^{23})^{-1}=R^{12}R^{13}$$

$$\tau(F):=RF^{21}e(-t)\text{ is an involution},\Phi_{\tau(F)}=(\Phi_F^{321})^{-1}$$

$$\mathrm{Sol}_0^\tau:=\{F:\tau(F)=F\}\text{ is non-empty; for }F\in\mathrm{Sol}_0^\tau,$$


Alekseev

$$e(t^{13}+t^{23})=\Phi^{213}e(t^{13})(\Phi^{231})^{-1}e(t^{23})\Phi^{321}$$

$$\text{and}\quad e(t^{12}+t^{13})=(\Phi^{132})^{-1}e(t^{13})\Phi^{312}e(t^{12})\Phi$$


Torossian

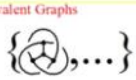
This is just a part of the Alekseev–Torossian work!

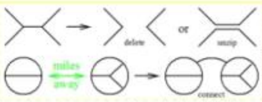
So What?

- Related to the Kashiwara-Vergne Conjecture!
- Will likely lead to an explicit tree-level associator, a linear equation away from a 1-loop equation, two linear equations away from a 2-loop associator, etc.!
- A baby version of the QUEA equations; we may be on the right tracks!

Knotted Trivalent Graphs

$\mathcal{O}(\triangle)=$





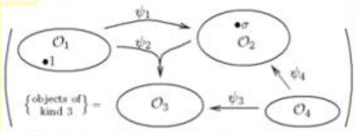
Theorem. KTG is generated by the unknotted $\triangle$ and the Möbius band, with identifiable relations between them.
Theorem. $Z(\triangle)$ is equivalent to an associator Φ.

Algebraic Knot Theory

Theorem. $\{\text{ribbon knots}\}\sim\{u\gamma\colon\gamma\in\mathcal{O}(\circ\!\circ),\ d\gamma=\bigcirc\bigcirc\}$.
Hence an expansion for KTG may tell us about ribbon knots, knots of genus 5, boundary links, etc.

"An Algebraic Structure"

$\mathcal{O}=$



- Has kinds, objects, operations, and maybe constants.
- Perhaps subject to some axioms.
- We always allow formal linear combinations.

Defining proj $\mathcal{O}$. The augmentation "ideal":

$$I=I_{\mathcal{O}}:=\left\{\begin{matrix}\text{formal differences of ob-}\\\text{jects “of the same kind”}\end{matrix}\right\}.$$

$$\text{Then }I^n:=\left\{\begin{matrix}\text{all outputs of algebraic}\\\text{expressions at least }n\text{ of}\\\text{whose inputs are in }I\end{matrix}\right\},\text{ and}$$

$$\mathrm{proj}\,\mathcal{O}:=\bigoplus_{n\geq0}I^n/I^{n+1}.$$

- Has same kinds and operations, but different objects and axioms.

Knot Theory Anchors.

- $(\mathcal{O}/I^{n+1})^*$ is "type n invariants".
- $(I^n/I^{n+1})^*$ is "weight systems".
- $\mathrm{proj}\,\mathcal{O}$ is $\mathcal{A}$, "chord diagrams".

Warmup Examples.

- The projectivization of a group is a graded associative algebra.
- A quandle: a set Q with a binary op $\wedge$ s.t.

$$1\wedge x=1,\quad x\wedge1=x\wedge x=x,\qquad\text{(appetizers)}$$
$$(x\wedge y)\wedge z=(x\wedge z)\wedge(y\wedge z).\qquad\text{(main)}$$

$L:=\mathrm{proj}\,Q$ is a graded Lie algebra: set $\bar{v}:=(v-1)$ (these generate I!), feed $1+\bar{x}$, $1+\bar{y}$, $1+\bar{z}$ in (main), and collect the surviving terms of lowest degree:

$$(\bar{x}\wedge\bar{y})\wedge\bar{z}=(\bar{x}\wedge\bar{z})\wedge\bar{y}+\bar{x}\wedge(\bar{y}\wedge\bar{z}).$$

An Expansion is $Z\colon\mathcal{O}\to\mathrm{proj}\,\mathcal{O}$ s.t. $Z(I^n)\subset(\mathrm{proj}\,\mathcal{O})_{\geq n}$ and $Z_{I^n/I^{n+1}}=Id_{I^n/I^{n+1}}$ (A "universal finite type invariant"). In practice, it is hard to determine $\mathrm{proj}\,\mathcal{O}$, but easy to guess a surjection $\rho:\mathcal{A}\to\mathrm{proj}\,\mathcal{O}$. So find $Z'\colon\mathcal{O}\to\mathcal{A}$ with $Z'(I^n)\subset\mathcal{A}_{\geq n}$ and $Z'_{I^n/I^{n+1}}\circ\rho_n=Id_{\mathcal{A}_n}$:

$$\begin{tikzcd}
\mathcal{O} \arrow[r, "Z'"] \arrow[dr, "Z"'] & \mathcal{A} \arrow[r, leftrightarrow] \arrow[d, "\rho"] & \mathcal{A}_n \\
 & \mathrm{proj}\,\mathcal{O} \arrow[r, leftrightarrow] & (\mathrm{proj}\,\mathcal{O})_n \arrow[u, "Z'_{I^n/I^{n+1}}"']
\end{tikzcd}$$

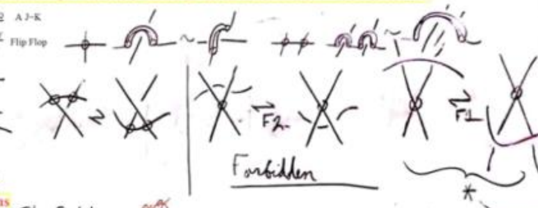
Can you make this diagram less confusing?

Homomorphic Expansions are expansions that intertwine the algebraic structure on $\mathcal{O}$ and $\mathrm{proj}\,\mathcal{O}$. They provide finite / combinatorial handles on global problems.


X-S Lin

Projectivization, Welded Knots and Alekseev–Torossian

Kauffman in Siegen



Circuit Algebras

* Have "circuits" with "ends"
* Can be wired arbitrarily.
* May have "relations" – de-Morgan, etc.

Shin Satoh

$S \cdot WK \longrightarrow$ Ribbon Torus Embeddings in $\mathbb{R}^4$

$\pi_1(K) \cong \pi_1(\mathbb{R}^4 - \Delta(K))$

"Welded trivalent (framed) tangles" are a circuit algebra:

"Welded Braids" are due to Fenn, Rimanyi and Rourke

$WT = \langle \ldots \rangle /$ R123, R4 (for vertices), F1.

Further operations: delete, unzip.

The "Chord Diagrams" — $\mathcal{A}_n^{wt}$. As we did for quandles, substitute ... into the various moves, to get relations. Also switch to "arrow diagram language". Get:

(tails commute)

R3 $\mapsto$ ... (4T really 6T)

R4 $\mapsto$ ... $= 0$ (vertex invariance)

The "Jacobi Diagrams" — $\mathcal{A}_n^{cc}$.

Theorem. (95%) $\mathcal{A}_n^{wt}$ is $\mathcal{A}_n^{cc}$ is $\mathcal{U}(\mathrm{tder}_n)$. (Pretend wheels die: $= 0$)

Here $\mathcal{A}_n^{cc}$ is $\{\ldots\}/\text{rels}$

trivalent directed trees with only 2-in 1-out vertices

In tensorland, this is "co-commutative Lie-bialgebras"

Rels: tails commute

Heads satisfy the only possible STU:

+ also IHX and vertex invariance

The Map $\alpha: \mathcal{A}_n^{tree} \to \mathcal{A}_n^{cc}$:

Theorem. (90%) α is an injection on $\mathcal{A}_n^{tree} \cong \mathcal{U}(\mathrm{sder}_n)$. Furthermore, there is a simple charactarization of im α, so we can tell "an arrowless element" when we see it.

Partial Dictionary.

$(R, F) \longleftrightarrow (\ldots)$ $\quad (r, t) \longleftrightarrow (\ldots)$

$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12} \longleftrightarrow$

$FF^{-1} = I \longleftrightarrow$ (unzip)

$F^{-1}e(x+y)F = e(x)e(y)$

$F^{23}R^{1,23} = R^{12}R^{13}F^{23} \longleftrightarrow$

$R^{12,3} = R^{13}R^{23} \longleftrightarrow$

$F^{12,3}R^{12,3} = R^{13}R^{23}F^{12,3}$ (unforbidding F1 makes this automatic)

$RF^{21}e(-t) = F \longleftrightarrow$

$\Phi = (F^{12,3})^{-1}(F^{1,2})^{-1}F^{2,13}F^{1,23} \longleftrightarrow$ = "The tetrahedron"

$\Phi \in \mathrm{sder} \longleftrightarrow$

The pentagon and the hexagons follow, with a minor twist, from the fact that we have an unzip-behaved invariant of KTG's.

The Main Theorem. (80%/0%) F's in Sol_0^r are in a bijective correspondance with tree-level associators for ordinary paranthesized tangles (or ordinary knotted trivalent graphs) / with homomorphic expansions for knotted welded trivalent tangles.

Disclaimer. Orientations, rotation numbers, framings, the vertical direction and the cyclic symmetry of the vertex may still make everything uglier. I hope not.

"God created the knots, all else in topology is the work of mortals"

Leopold Kronecker (paraphrased)

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