Pensieve header: Proof of invariance of ρ_1.

In[•]:=
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SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\Waco-2203"];
Once[<< KnotTheory`; << RVK.m];
δ_i_,j_ := If[TrueQ[i == j], 1, 0];
ρ[K_] := Module[{Cs, r, n, A, s, i, j, k, Δ, G, g, ρ1},
  {Cs, r} = List @@ RVK[K]; n = Length[Cs];
  A = IdentityMatrix[2 n + 1];
  Cases[Cs, {s_, i_, j_} :> (A[[{i, j}, {i + 1, j + 1}]] = ({{-T^s, T^s - 1}, {0, -1}}))];
  Δ = T^(-Total[r] - Total[Cs[[All, 1]]])/2 Det[A];
  G = Inverse[A]; g_α_,β_ := G[[α, β]];
  Echo@Union@Flatten@Simplify@Table[{s, i, j} = c;
    {g_ik == δ_ik + T^s g_i+1,k + (1 - T^s) g_j+1,k, g_jk == δ_jk + g_j+1,k,
     g_ki == T^-s (g_k,i+1 - δ_k,i+1), g_kj == g_k,j+1 - (1 - T^s) g_ki - δ_k,j+1},
    {c, Cs}, {k, 2 n + 1}
   ];
  ρ1 = Sum_k=1^n ({s, i, j} = Cs[[k]];
    s ((1 - T^s) g_ji (g_ji - g_ii) + 2 g_jj g_ji - g_ji g_ij - g_jj g_ii - g_ji + g_ii - 1 / 2));
  ρ1 -= Sum_k=1^2 n r[[k]] (g_kk - 1 / 2);
  Factor@{Δ, Δ^2 ρ1}];
Table[K → ρ[K], {K, AllKnots[{3, 6}]}]
```

» {True}

» {True}

» {True}

» {True}

» {True}

» {True}

» {True}

Out[•]=
$$\left\{\text{Knot}[3,1]\to\left\{\frac{1-T+T^2}{T},\ \frac{(-1+T)^2\left(1+T^2\right)}{T^2}\right\},\ \text{Knot}[4,1]\to\left\{-\frac{1-3T+T^2}{T},\ 0\right\},\right.$$
$$\text{Knot}[5,1]\to\left\{\frac{1-T+T^2-T^3+T^4}{T^2},\ \frac{(-1+T)^2\left(1+T^2\right)\left(2+T^2+2T^4\right)}{T^4}\right\},$$
$$\text{Knot}[5,2]\to\left\{\frac{2-3T+2T^2}{T},\ \frac{(-1+T)^2\left(5-4T+5T^2\right)}{T^2}\right\},$$
$$\text{Knot}[6,1]\to\left\{-\frac{(-2+T)(-1+2T)}{T},\ \frac{(-1+T)^2\left(1-4T+T^2\right)}{T^2}\right\},$$
$$\text{Knot}[6,2]\to\left\{-\frac{1-3T+3T^2-3T^3+T^4}{T^2},\ \frac{(-1+T)^2\left(1-4T+4T^2-4T^3+4T^4-4T^5+T^6\right)}{T^4}\right\},$$
$$\left.\text{Knot}[6,3]\to\left\{\frac{1-3T+5T^2-3T^3+T^4}{T^2},\ 0\right\}\right\}$$

In[•]:=
```
PR[s_, i_, j_] := {
  g_{im_} :> δ_{im} + T^s g_{i+1,m} + (1 - T^s) g_{j+1,m},  g_{jm_} :> δ_{jm} + g_{j+1,m},
  g_{m_,i} :> T^-s (g_{m,i+1} - δ_{m,i+1}),  g_{m_j} :> g_{m,j+1} - (1 - T^s) g_{mi} - δ_{m,j+1}
 }
```

In[•]:=
```
R[s_, i_, j_] := s ((1 - T^s) g_{ji} (g_{ji} - g_{ii}) + 2 g_{jj} g_{ji} - g_{ji} g_{ij} - g_{jj} g_{ii} - g_{ji} + g_{ii} - 1 / 2)
```

In[•]:=
```
lhs = Simplify[R[1, i, j] + R[1, i + 1, k] + R[1, j + 1, k + 1] //.
    PR[1, i, j] ⋃ PR[1, i + 1, k] ⋃ PR[1, j + 1, k + 1]]
```

Out[•]=
$$-\frac{1}{2T^2}\left(-2(-1+T)T g_{2+j,2+i}^2+\right.$$
$$2g_{2+j,2+i}\left(T^2+T^2g_{2+i,2+j}-2T^2g_{2+j,2+j}+g_{2+k,2+i}-2Tg_{2+k,2+i}+T^2g_{2+k,2+i}-Tg_{2+k,2+j}+T^2g_{2+k,2+j}\right)+$$
$$2g_{2+i,2+i}\left(-2T^2+(-1+T)Tg_{2+j,2+i}+T^2g_{2+j,2+j}-g_{2+k,2+i}+Tg_{2+k,2+i}+T^2g_{2+k,2+k}\right)+$$
$$T\left(3T-2(-1+T)g_{2+k,2+i}^2+2Tg_{2+k,2+j}+2Tg_{2+j,2+k}g_{2+k,2+j}+2g_{2+k,2+j}^2-\right.$$
$$2Tg_{2+k,2+j}^2+2g_{2+j,2+j}\left((-1+T)g_{2+k,2+i}+(-1+T)g_{2+k,2+j}+T(-1+g_{2+k,2+k})\right)-$$
$$\left.\left.4Tg_{2+k,2+j}g_{2+k,2+k}+2g_{2+k,2+i}\left(T+Tg_{2+i,2+k}-2(-1+T)g_{2+k,2+j}-2Tg_{2+k,2+k}\right)\right)\right)$$

In[•]:=
```
PR[1, i, j]
```

Out[•]=
$$\left\{g_{i,m\$\_}:>\delta_{i,m\$}+T^1g_{i+1,m\$}+\left(1-T^1\right)g_{j+1,m\$},\ g_{j,m\$\_}:>\delta_{j,m\$}+g_{j+1,m\$},\right.$$
$$\left.g_{m\$\_,i}:>T^{-1}\left(g_{m\$,i+1}-\delta_{m\$,i+1}\right),\ g_{m\$\_,j}:>g_{m\$,j+1}-\left(1-T^1\right)g_{m\$,i}-\delta_{m\$,j+1}\right\}$$

```
In[•]:= rhs = Simplify[R[1, j, k] + R[1, i, k + 1] + R[1, i + 1, j + 1] //.
            PR[1, j, k] ⋃ PR[1, i, k + 1] ⋃ PR[1, i + 1, j + 1]]
```

$$\text{Out}[\bullet]= -\frac{1}{2\,T^2}\Big(-2\,(-1+T)\,T\,g_{2+j,2+i}^2+2\,g_{2+j,2+i}\Big(T^2+T^2\,g_{2+i,2+j}-2\,T^2\,g_{2+j,2+j}+g_{2+k,2+i}-2\,T\,g_{2+k,2+i}+T^2\,g_{2+k,2+i}-T\,g_{2+k,2+j}+T^2\,g_{2+k,2+j}\Big)+2\,g_{2+i,2+i}\Big(-2\,T^2+(-1+T)\,T\,g_{2+j,2+i}+T^2\,g_{2+j,2+j}-g_{2+k,2+i}+T\,g_{2+k,2+i}+T^2\,g_{2+k,2+k}\Big)+T\Big(3\,T-2\,(-1+T)\,g_{2+k,2+i}^2+2\,T\,g_{2+k,2+j}+2\,T\,g_{2+j,2+k}\,g_{2+k,2+j}+2\,g_{2+k,2+j}^2-2\,T\,g_{2+k,2+j}^2+2\,g_{2+j,2+j}\Big((-1+T)\,g_{2+k,2+i}+(-1+T)\,g_{2+k,2+j}+T\,(-1+g_{2+k,2+k})\Big)-4\,T\,g_{2+k,2+j}\,g_{2+k,2+k}+2\,g_{2+k,2+i}\Big(T+T\,g_{2+i,2+k}-2\,(-1+T)\,g_{2+k,2+j}-2\,T\,g_{2+k,2+k}\Big)\Big)\Big)$$

```
In[•]:= Simplify[lhs - rhs]
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$$\text{Out}[\bullet]= 0$$