UNC-1610 handout on Oct 7

October 7, 2016 9:38 AM

Dror Bar-Natan: Talks: UNC-1610:

ωεβ:=http://drorbn.net/UNC-1610/

Work in Progress!

A Poly-Time Knot Polynomial Via Solvable Approximation

Abstract. I will construct the first poly-time-computable knot polynomial since Alexander's [Al, 1928] by using some new commutator-calculus techniques and a Lie algebra $\mathfrak{g}_1$ which is at the same time solvable and an approximation of the simple Lie algebra sl_2.

Expected! Finite-type invariants include all coefficients of all quantum knot polynomials (appropriately parametrized), and each is computable in poly-time. Yet

A green paradise...

d	2	3	4	5	6	7	8	...
known f.t. invts in $O(n^d)$	1	1	∞	3	4	8	11	...

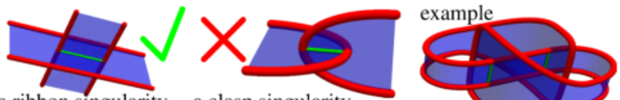
This is an unreasonable picture! So there ought to be further poly-time polynomial invariants.

Also. • The line above the Alexander line in the Melvin-Morton [MM, Ro] expansion of the coloured Jones polynomial. • The 2-loop contribution to the Kontsevich integral.

Rozansky

Paradise! Foremost reason: *OBVIOUSLY.* Cf. proving (incomputable A)=(incomputable B), or categorifying (incomputable C).

Secondary reason: may disprove {ribbon} = {slice}:



a ribbon singularity a clasp singularity

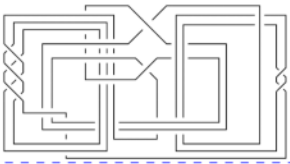
A bit about ribbon knots. A "ribbon knot" is a knot that can be presented as the boundary of a disk that has "ribbon singularities", but no "clasp singularities". A "slice knot" is a knot in $S^3 = \partial B^4$ which is the boundary of a non-singular disk in B^4. Every ribbon knots is clearly slice, yet,

Conjecture. Some slice knots are not ribbon.

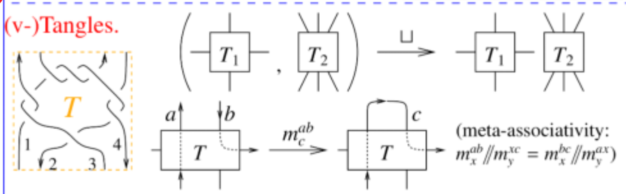
In [BN2] I list 5 criteria an invariant needs to meet to have a fair chance of detecting non-ribbons. Ours meets all 5.

more to end 2

[GST]: Gompf, Scharlemann, Thompson:



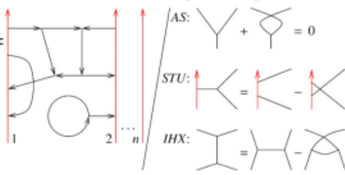
(v-)Tangles.



(meta-associativity: $m_x^{ab} /\!/ m_y^{xc} = m_x^{bc} /\!/ m_y^{ax}$)

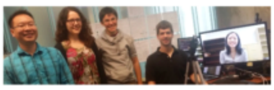
Theorem [EK, Ha, En, Se]. There is a "homomorphic expansion"

$$Z: \left\{\begin{matrix} S\text{-component} \\ (v/b\text{-})\text{tangles}\end{matrix}\right\} \to \mathcal{A}^v_S :=$$



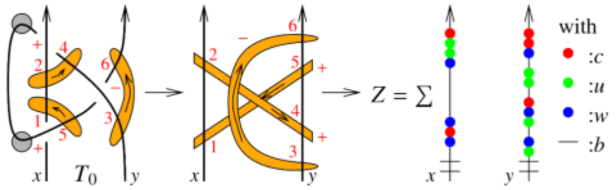
Etingof Kazhdan Haviv Enriquez Ševera

Videos of a 4-hour version of this talk are at ωεβ/LD. Videos of private seminar meetings are at ωεβ/PP.



Many thanks: Vo, Halacheva, Dalvit, Ens, Lee (van der Veen, Schaveling)

Algebras and Invariants. Given any unital algebra A (typically, $A \sim \hat{\mathcal{U}}(\mathfrak{g})$), appropriate orange $R \in A \otimes A$, and appropriate cuaps $\in A$, get an $A^{\otimes S}$-valued invariant of pure S-component tangles:



Good News. In theory, enough to know R, the cuaps, and stitching/multiplication $m_k^{ij}: A_i \otimes A_j \to A_k$.

Problem. Extract information out of Z.

Textbook Solution. Use representation theory ...works, slowly.

Today's Solution. (with van der Veen) For some specific $\mathfrak{g}$'s, work in a space of "formulas of a specific type" for elements of $\hat{\mathcal{U}}(\mathfrak{g})^{\otimes S}$:

$$\left\{\begin{matrix}\text{ordered perturbed} \\ \text{Gaussian formulas}\end{matrix}\right\} \to \hat{\mathcal{U}}(\mathfrak{g})^{\otimes S}$$

van der Veen

1-Smidgen sl_2 Let $\mathfrak{g}_1$ be the 4-dimensional Lie algebra $\mathfrak{g}_1 = \langle b, c, u, w\rangle$ over the ring $R = \mathbb{Q}[\epsilon]/(\epsilon^2 = 0)$, with b central and with $[w, c] = w$, $[c, u] = u$, and $[u, w] = b - 2\epsilon c$, with CYBE $r_{ij} = (b_i - \epsilon c_i)c_j + u_i w_j$ in $\mathcal{U}(\mathfrak{g}_1)^{\otimes\{i,j\}}$. Over $\mathbb{Q}$, $\mathfrak{g}_1$ is a solvable approximation of sl_2: $\mathfrak{g}_1 \supset \langle b, u, w, \epsilon b, \epsilon c, \epsilon u, \epsilon w\rangle \supset \langle b, \epsilon b, \epsilon c, \epsilon u, \epsilon w\rangle \supset 0$. (note: $\deg(b, c, u, w, \epsilon) = (1, 0, 1, 0, 1)$)

0-Smidgen sl_2 ☺. Let $\mathfrak{g}_0$ be $\mathfrak{g}_1$ at $\epsilon = 0$, or $\mathbb{Q}\langle b, c, u, w\rangle/([b, \cdot] = 0, [c, u] = u, [c, w] = -w, [u, w] = b$ with $r_{ij} = b_i c_j + u_i w_j$. It is $\mathfrak{b}^* \rtimes \mathfrak{b}$ where $\mathfrak{b}$ is the 2D Lie algebra $\mathbb{Q}\langle c, w\rangle$ and (b, u) is the dual basis of (c, w). It is even more valuable than $\mathfrak{g}_1$, but topology already got by other means almost everything $\mathfrak{g}_0$ has to give.

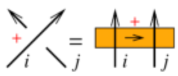
How did these arise? $sl_2 = \mathfrak{b}^+ \oplus \mathfrak{b}^-/\mathfrak{h} =: sl_2^+/\mathfrak{h}$, where $\mathfrak{b}^+ = \langle c, w\rangle/[w, c] = w$ is a Lie bialgebra with $\delta: \mathfrak{b}^+ \to \mathfrak{b}^+ \otimes \mathfrak{b}^+$ by $\delta: (c, w) \mapsto (0, c \wedge w)$. Going back, $sl_2^+ = \mathcal{D}(\mathfrak{b}^+) = (\mathfrak{b}^+)^* \oplus \mathfrak{b}^+ = \langle b, u, c, w\rangle/\cdots$. **Idea.** Replace $\delta \to \epsilon\delta$ over $\mathbb{Q}[\epsilon]/(\epsilon^{k+1} = 0)$. At $k = 0$, get $\mathfrak{g}_0$. At $k = 1$, get $[w, c] = w$, $[w, b'] = -\epsilon w$, $[c, u] = u$, $[b', u] = -\epsilon u$, $[b', c] = 0$, and $[u, w] = b' - \epsilon c$. Now note that $b' + \epsilon c$ is central, so switch to $b := b' + \epsilon c$. This is $\mathfrak{g}_1$.

Ordering Symbols. $\mathbb{O}(poly \mid specs)$ plants the variables of *poly* in $S(\oplus_i \mathfrak{g})$ on several tensor copies of $\mathcal{U}(\mathfrak{g})$ according to *specs*. E.g.,

$$\mathbb{O}\left(c_1^3 u_1 c_2 e^{u_3} w_3^9 | x: w_3 c_1,\ y: u_1 u_3 c_2\right) = w^9 c^3 \otimes u e^u c \in \mathcal{U}(\mathfrak{g})_x \otimes \mathcal{U}(\mathfrak{g})_y$$

This enables the description of elements of $\hat{\mathcal{U}}(\mathfrak{g})^{\otimes S}$ using commutative polynomials / power series.

0-Smidgen Invariants. $r = Id \in \mathfrak{b}^- \otimes \mathfrak{b}^+$ solves the CYBE $[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$ in $\mathcal{U}(\mathfrak{g}_0)^{\otimes 3}$ and, by luck,



$$= R_{ij} = e^{r_{ij}} = e^{b_i c_j + u_i w_j} \in \mathcal{U}(\mathfrak{g}_{0,i} \oplus \mathfrak{g}_{0,j})$$

solves YB/R3.

Lemma. $R_{ij} = e^{b_i c_j + u_i w_j} = \mathbb{O}\left(\exp\left(b_i c_j + \frac{e^{b_i}-1}{b_i} u_i w_j\right) | i: u_i,\ j: c_j w_j\right)$

Example. $Z(T_0) = \qquad = \sum_{m,n} \frac{b_i^{m-n}(e^{b_i}-1)^n}{m!n!} u^n \otimes c^m w^n.$

$$\mathbb{O}\Bigg(\exp\left(b_5 c_1 + \frac{e^{b_5}-1}{b_5} u_5 w_1 + b_2 c_4 + \frac{e^{b_2}-1}{b_2} u_2 w_4 - b_3 c_6 + \frac{e^{-b_3}-1}{b_3} u_3 w_6\right) \Big| \; x: c_1 w_1 u_2,\ y: u_3 c_4 w_4 u_5 c_6 w_6\Bigg) = \mathbb{O}\left(? | x: c_x u_x w_x,\ y: c_y u_y w_y\right)$$

"cuw form"

Goal. Write ? as a Gaussian: ωe^{L+Q} where L bilinear in b_i and c_i with integer coefficients, Q a balanced quadratic in u_i and w_i with coefficients in $R_S := \mathbb{Q}(b_i, e^{b_i})$, and $\omega \in R_S$.

The Big $\mathfrak{g}_0$ Lemma. Under $[c,u]=u$, $[c,w]=-w$, and $[u,w]=b$:

1. $N_k^{c_i c_j} := \mathbb{O}(\zeta|c_i c_j) \stackrel{\equiv}{\to} \mathbb{O}(\zeta/(c_i, c_j \to c_k)|c_k)$ (Meaning, $N_k^{c_i c_j}: \zeta \mapsto (\zeta/(c_i, c_j \to c_k))$ and the diagram commutes. Trivial, also for b, u, w.)

$$\begin{tikzcd} S \arrow[r, "N_k^{c_i c_j}"] \arrow[dr, "\mathbb{O}(-|c_i c_j)"'] & S \arrow[d, "\mathbb{O}(-|c_k)"] \\ & \mathcal{U} \end{tikzcd}$$

2a. $N^{uc} := \mathbb{O}(e^{\gamma c+\beta u}|uc) \stackrel{\equiv}{\to} \mathbb{O}(e^{\gamma c+e^{-\gamma}\beta u}|cu)$ (means $e^{\beta u}e^{\gamma c} = e^{\gamma c}e^{e^{-\gamma}\beta u}$

2b. $N^{wc} := \mathbb{O}(e^{\gamma c+\alpha w}|wc) \stackrel{\equiv}{\to} \mathbb{O}(e^{\gamma c+e^{\gamma}\alpha w}|cw)$ …in the $\{ax+b\}$ group)

3. $\mathbb{O}(e^{\alpha w+\beta u}|wu) = \mathbb{O}(e^{-b\alpha\beta+\alpha w+\beta u}|uw)$ (the Weyl relations)

4. $\mathbb{O}(e^{\delta uw}|wu)e^{\beta u} = e^{\nu\beta u}\mathbb{O}(e^{\delta uw}|wu)$, with $\nu = (1+b\delta)^{-1}$

(a. expand and crunch. b. use $w = b\hat{x}$, $u = \partial_x$. c. use "scatter and glow".)

5. $\mathbb{O}(e^{\delta uw}|wu) = \mathbb{O}(\nu e^{\nu\delta uw}|uw)$ (same techniques)

6. $N^{wu} := \mathbb{O}(e^{\beta u+\alpha w+\delta uw}|wu) \stackrel{\equiv}{\to} \mathbb{O}(\nu e^{-b\nu\alpha\beta+\nu\alpha w+\nu\beta u+\nu\delta uw}|uw)$

Sneaky. α may contain (other) u's, β may contain (other) w's.

Strand Stitching, m_k^{ij}, is defined as the composition

$$c_i u_i \overline{w_i c_j}\, u_j w_j \xrightarrow{N_k^{w_i c_j}} c_i\, \overline{u_i c_k}\, \overline{w_k u_j}\, w_j \xrightarrow{N_k^{u_i c_k} /\!/ N_k^{w_k u_j}} \overline{c_i c_k}\, \overline{u_k u_k}\, \overline{w_k w_j}$$

 $\xrightarrow{N_k^{c_i c_k} /\!/ - - /\!/ N_k^{w_k w_j}} c_k u_k w_k$

Γ-calculus [BNS, BN1]. After re-packaging, especially setting $t_i := e^{b_i}$, an S-component tangle T has $\Gamma(T) \in R_S \times M_{S\times S}(R_S) = \left\{\begin{array}{c|c} \omega & S \\ \hline S & A \end{array}\right\}$ with $R_S := \mathbb{Z}(\{t_a : a \in S\})$:

$$({}_a\nearrow_b, {}_b\nwarrow_a) \to \begin{array}{c|cc} 1 & a & b \\ \hline a & 1 & 1-t_a^{\pm 1} \\ b & 0 & t_a^{\pm 1} \end{array} \qquad T_1 \sqcup T_2 \to \begin{array}{c|cc} \omega_1\omega_2 & S_1 & S_2 \\ \hline S_1 & A_1 & 0 \\ S_2 & 0 & A_2 \end{array}$$

$$\begin{array}{c|ccc} \omega & a & b & S \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi \end{array} \xrightarrow[t_a, t_b \to t_c]{m_c^{ab}} \left(\begin{array}{c|cc} (1-\beta)\omega & c & S \\ \hline c & \gamma + \frac{\alpha\delta}{1-\beta} & \epsilon + \frac{\delta\theta}{1-\beta} \\ S & \phi + \frac{\alpha\psi}{1-\beta} & \Xi + \frac{\psi\theta}{1-\beta} \end{array}\right)$$

For long knots, ω is Alexander, and that's the fastest Alexander algorithm I know!

Dunfield: 1000-crossing fast.

1-Smidgen Invariants. Much is the same:

The Big $\mathfrak{g}_1$ Lemma. Parts 1 and 2 are the same, yet

6. $\mathbb{O}\left(e^{\alpha w+\beta u+\delta uw}|wu\right) = \mathbb{O}\left(\nu(1+\epsilon\nu\Lambda)e^{\nu(-b\alpha\beta+\alpha w+\beta u+\delta uw)}|cuw\right)$

Here Λ is for Λόγος, "a principle of order and knowledge", a balanced quartic in α, β, c, u, and w:

$$\begin{aligned}\Lambda = &-b\nu\left(\nu^2\alpha^2\beta^2 + 4\delta\nu\alpha\beta + 2\delta^2\right)/2 - \delta\nu^3(3b\delta+2)\beta^2u^2/2 \\ &- b\delta^4\nu^3u^2w^2/2 - \delta^2\nu^3(2b\delta+1)\beta u^2 w \\ &- \nu^2(2b\delta+1)(\nu\alpha\beta+2\delta)\beta u - 2b\delta^2\nu^2(\nu\alpha\beta+\delta)uw \\ &+ \delta\nu^3(b\delta+2)\alpha^2w^2/2 + 2(\nu\alpha\beta+\delta)c + 2\delta\nu\beta cu + 2\delta^2\nu cuw \\ &+ 2\delta\nu\alpha cw + \delta^2\nu^3\alpha uw^2 + \nu^2(\nu\alpha\beta+2\delta)\alpha w.\end{aligned}$$

Proof. A brutal hell.

Problem. We now need to normal-order perturbed Gaussians!

Solution. Borrow some tactics from QFT:

$$\mathbb{O}(\epsilon P(c,u)e^{\gamma c+\beta u}|uc) = \mathbb{O}(\epsilon P(\partial_\gamma, \partial_\beta)e^{\gamma c+\beta u}|uc) = \mathbb{O}(\epsilon P(\partial_\gamma, \partial_\beta)e^{\gamma c+e^{-\gamma}\beta u}|cu),$$

and likewise

$$\mathbb{O}\left(\epsilon P(u,w)e^{\alpha w+\beta u+\delta uw}|wu\right) = \mathbb{O}\left(\epsilon P(\partial_\beta, \partial_\alpha)\nu e^{\nu(-b\alpha\beta+\alpha w+\beta u+\delta uw)}|cuw\right)$$

Note. Strand stitching requires a tiny extra step.

Finally, the values of the generators ⤫, ⤬, $\overrightarrow{n}$, $\overleftarrow{n}$, $\underset{\rightarrow}{u}$, and $\underset{\leftarrow}{u}$, are set by brutally solving many equations, non-uniquely.

Pragmatic Simplifications. Get rid of $\zeta = (e^b-1)/b$ factors by rescaling $u \to \bar{u} = \zeta u$. Complement this with $\beta \to \bar{\beta} = \zeta^{-1}\beta$, $\delta \to \bar{\delta} = \zeta^{-1}\delta$, $\epsilon \to \bar{\epsilon} = \zeta^{-1}\epsilon$. Simplify further by naming $e^b \to t$; e.g., $\nu \to \bar{\nu} = (1+(t-1)\delta)^{-1}$. Get confused by renaming $(\bar{u}, \bar{\beta}, \bar{\delta}, \bar{\nu}) \to (u, \beta, \delta, \nu)$, and more confused by working with $\mu = \nu^{-1}$ and $\mathbb{E}(\omega, L, Q, P) := \omega^{-1}(1+\epsilon\omega^{-4}P)e^{L+\omega^{-1}Q}$, where $\omega \in R := \mathbb{Q}(t_k)$, $L = \sum l_{ij}b_ic_j$ with $l_{ij} \in \mathbb{Z}$, $Q = \sum q_{ij}u_iw_j$ with $q_{ij} \in R$, and P is a balanced quartic polynomial in c_i, u_i, and w_i with coefficients in R. Magically, all coefficients are now Laurent polynomials in the t_k's.

Rough complexity estimate, after $t_k \to t$. n: xing number; w: width, maybe $\sim\sqrt{n}$.

$$\underbrace{n}_{A} \overbrace{\sum_{d=0}^{4} \underbrace{w^{4-d}}_{E} \underbrace{w^d}_{F} \underbrace{n^2}_{G}}^{B} = n^3w^4 \in [n^5, n^7]$$

A: go over stitchings in order. B: multiplication ops per $N^{u_iw_j}$. d: deg of u_i, w_j in P. E: #terms of deg d in P. F: ops per term. G: cost per polynomial multiplication op.

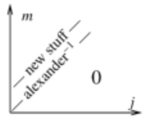
Expectation. Our invariant is the "1-higher diagonal" in the MMR expansion of the coloured Jones polynomial J_λ.

Melvin, Morton, Garoufalidis

Theorem ([BNG], conjectured [MM], elucidated [Ro]). Let $J_d(K)$ be the coloured Jones polynomial of K, in the d-dimensional representation of $sl(2)$. Writing

$$\left.\frac{(q^{1/2}-q^{-1/2})J_d(K)}{q^{d/2}-q^{-d/2}}\right|_{q=e^\hbar} = \sum_{j,m\geq 0} a_{jm}(K)d^j\hbar^m,$$

"below diagonal" coefficients vanish, $a_{jm}(K) = 0$ if $j > m$, and "on diagonal" coefficients give the inverse of the Alexander polynomial: $\left(\sum_{m=0}^\infty a_{mm}(K)\hbar^m\right)\cdot A(K)(e^\hbar) = 1$.



I mean business!

The 6ST48 Knot:

Demo Programs for 0-Co. — ωεβ/Demo

The R-matrices

$$R^+_{0,i\_,j\_} := \mathbb{E}\left[b_i c_j + b_i^{-1}\left(e^{b_i}-1\right) u_i w_j\right];$$
$$R^-_{0,i\_,j\_} := \mathbb{E}\left[-b_i c_j + b_i^{-1}\left(e^{-b_i}-1\right) u_i w_j\right];$$

Utilities

$$\text{CF}[\omega\_ . \mathbb{E}[Q\_]] := \text{Simplify}[\omega]\,\mathbb{E}[\text{Simplify}[Q]];$$
$$\mathbb{E} \;/\!:\; \mathbb{E}[Q1\_]\,\mathbb{E}[Q2\_] := \text{CF@}\mathbb{E}[Q1+Q2];$$
$$\omega1\_ . \mathbb{E}[Q1\_] \equiv \omega2\_ . \mathbb{E}[Q2\_] := \text{Simplify}[\omega1 == \omega2 \wedge Q1 == Q2];$$

Normal Ordering Operators

$$N_{u_{i\_}c_{j\_}\to k\_}[\omega\_ . \mathbb{E}[Q\_]] := \text{CF}\Big[$$
$$\quad \omega\,\mathbb{E}[e^{-\gamma}\beta u_k + \gamma c_k + (Q \;/.\; c_j \,|\, u_i \to 0)] \;/.\; \{\gamma \to \partial_{c_j}Q,\ \beta \to \partial_{u_i}Q\}\Big];$$
$$N_{w_{i\_}c_{j\_}\to k\_}[\omega\_ . \mathbb{E}[Q\_]] := \text{CF}\Big[$$
$$\quad \omega\,\mathbb{E}[e^{\gamma}\alpha w_k + \gamma c_k + (Q \;/.\; c_j \,|\, w_i \to 0)] \;/.\; \{\gamma \to \partial_{c_j}Q,\ \alpha \to \partial_{w_i}Q\}\Big];$$
$$N_{w_{i\_}u_{j\_}\to k\_}[\omega\_ . \mathbb{E}[Q\_]] := \text{CF}\Big[$$
$$\quad \nu\,\omega\,\mathbb{E}[-b_k\nu\alpha\beta + \nu\beta u_k + \nu\delta u_k w_k + \nu\alpha w_k + (Q \;/.\; w_i \,|\, u_j \to 0)] \;/.$$
$$\quad\quad \nu \to (1+b_k\delta)^{-1} \;/.$$
$$\quad\quad \{\alpha \to \partial_{w_i}Q \;/.\; u_j \to 0,\ \beta \to \partial_{u_j}Q \;/.\; w_i \to 0,\ \delta \to \partial_{w_i,u_j}Q\}\Big];$$

Stitching

$$m_{i\_,j\_\to k\_}[\omega\_ . \mathbb{E}[Q\_]] := \text{CF}\Big[\text{Module}\big[\{x\},$$
$$\quad \left(\omega\,\mathbb{E}[Q] \;/.\; b_{i|j} \to b_k \;//\; N_{w_i c_j\to x} \;//\; N_{u_i c_x\to x} \;//\; N_{w_x u_j\to x}\right) \;/.$$
$$\quad\quad \{c_i \to c_k,\ w_j \to w_k,\ y\_{}_x :\to y_k\}\big]\Big]$$

Some calculations for T_0

$$T_{0,0} = R^+_{0,5,1}\,R^+_{0,2,4}\,R^-_{0,3,6}$$
$$\mathbb{E}\left[b_5 c_1 + b_2 c_4 - b_3 c_6 + \frac{\left(-1+e^{b_5}\right)u_5 w_1}{b_5} + \frac{\left(-1+e^{b_2}\right)u_2 w_4}{b_2} + \frac{\left(-1+e^{-b_3}\right)u_3 w_6}{b_3}\right]$$
$$T_{0,1} = T_{0,0} \;//\; N_{u_3 c_4\to 4}$$
$$\mathbb{E}\Big[$$
$$b_5 c_1 + b_2 c_4 - b_3 c_6 + \frac{\left(-1+e^{b_5}\right)u_5 w_1}{b_5} + \frac{\left(-1+e^{b_2}\right)u_2 w_4}{b_2} + \frac{e^{-b_2}\left(-1+e^{-b_3}\right)u_4 w_6}{b_3}\Big]$$
$$T_{0,2} = T_{0,1} \;//\; N_{w_4 u_5\to 4}$$
$$\mathbb{E}\Big[b_5 c_1 + b_2 c_4 + \frac{\left(-1+e^{b_5}\right)\left(-\left(-1+e^{b_2}\right)b_4 u_2 + b_2 u_4\right)w_1}{b_2 b_5} +$$
$$\frac{\left(-1+e^{b_2}\right)u_2 w_4}{b_2} - \frac{b_3^2 c_6 + e^{-b_2-b_3}\left(-1+e^{b_3}\right)u_4 w_6}{b_3}\Big]$$
$$T_{0,2} \;//\; N_{w_1 u_2\to 1}$$
$$\frac{1}{1-\frac{\left(-1+e^{b_2}\right)\left(-1+e^{b_5}\right)b_1 b_4}{b_2 b_5}}\,\mathbb{E}\Big[\frac{1}{b_3\left(\left(-1+e^{b_2}\right)\left(-1+e^{b_5}\right)b_1 b_4 - b_2 b_5\right)}$$
$$\Big(b_3 b_5\left(\left(-1+e^{b_2}\right)\left(-1+e^{b_5}\right)b_1 b_4 - b_2 b_5\right)c_1 +$$
$$b_2 b_3\left(\left(-1+e^{b_2}\right)\left(-1+e^{b_5}\right)b_1 b_4 - b_2 b_5\right)c_4 +$$
$$\left(-1+e^{b_2}\right)\left(-1+e^{b_5}\right)b_3 b_4 u_1 w_1 - \left(-1+e^{b_5}\right)b_2 b_3 u_4 w_1 -$$
$$\left(-1+e^{b_2}\right)b_3 b_5 u_1 w_4 + \left(-1+e^{b_2}\right)\left(-1+e^{b_5}\right)b_1 b_3 u_4 w_4 -$$
$$\left(\left(-1+e^{b_2}\right)\left(-1+e^{b_5}\right)b_1 b_4 - b_2 b_5\right)$$
$$\left(b_3^2 c_6 + e^{-b_2-b_3}\left(-1+e^{b_3}\right)u_4 w_6\right)\Big)\Big]$$
$$T_{0,0} \;//\; m_{1,2\to 1} \;//\; m_{3,4\to 3} \;//\; m_{3,5\to 3} \;//\; m_{3,6\to 3}$$
$$\frac{1}{1-\left(-1+e^{b_1}\right)\left(-1+e^{b_3}\right)}\,\mathbb{E}\Big[b_3 c_1 + b_1 c_3 - b_3 c_3 +$$
$$\frac{\left(-1+e^{b_1}\right)\left(-1+e^{b_3}\right)u_1 w_1}{\left(-e^{b_1}-e^{b_3}+e^{b_1+b_3}\right)b_1} - \frac{e^{-b_3}\left(-1+e^{b_1}\right)\left(b_3 u_1 - e^{b_3}\left(-1+e^{b_3}\right)b_1 u_3\right)w_3}{\left(-e^{b_1}-e^{b_3}+e^{b_1+b_3}\right)b_1 b_3} +$$
$$\frac{e^{-b_1}\left(-1+e^{b_3}\right)u_3\left(-e^{b_1+b_3}w_1 + \left(e^{b_1}+e^{b_3}-e^{b_1+b_3}\right)w_3\right)}{\left(-e^{b_1}-e^{b_3}+e^{b_1+b_3}\right)b_3}\Big]$$

Verifying meta-associativity

$$Q0 = \mathbb{E}[\text{Sum}[f_i c_i, \{i,3\}] + \text{Sum}[f_{i,j} u_i w_j, \{i,3\}, \{j,3\}]]$$
$$\mathbb{E}[c_1 f_1 + c_2 f_2 + c_3 f_3 + u_1 w_1 f_{1,1} + u_1 w_2 f_{1,2} + u_1 w_3 f_{1,3} + u_2 w_1 f_{2,1} +$$
$$u_2 w_2 f_{2,2} + u_2 w_3 f_{2,3} + u_3 w_1 f_{3,1} + u_3 w_2 f_{3,2} + u_3 w_3 f_{3,3}]$$
$$(Q0 \;//\; m_{1,2\to 1} \;//\; m_{1,3\to 1}) \equiv (Q0 \;//\; m_{2,3\to 2} \;//\; m_{1,2\to 1})$$

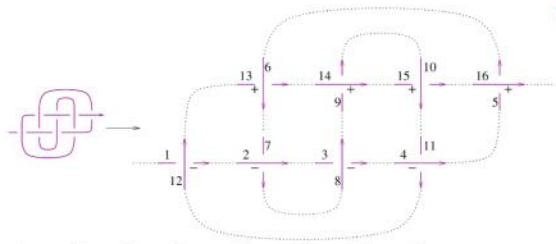
True

Testing R3

$$t1 = R^+_{0,1,2}\,R^+_{0,3,4}\,R^+_{0,5,6} \;//\; m_{3,5\to x} \;//\; m_{1,6\to y} \;//\; m_{2,4\to z}$$
$$\mathbb{E}\left[b_x(c_y+c_z) + \frac{\left(-1+e^{b_x}\right)u_x(w_y+w_z)}{b_x} + \frac{b_y^2 c_z + \left(-1+e^{b_y}\right)u_y w_z}{b_y}\right]$$
$$t1 \equiv (R^+_{0,1,2}\,R^+_{0,3,4}\,R^+_{0,5,6} \;//\; m_{1,3\to x} \;//\; m_{2,5\to y} \;//\; m_{4,6\to z})$$

True

8_{17}



$$z1 = R^-_{0,12,1}\,R^-_{0,2,7}\,R^-_{0,8,3}\,R^-_{0,4,11}\,R^+_{0,16,5}\,R^+_{0,6,13}\,R^+_{0,14,9}\,R^+_{0,10,15};$$
$$\text{Do}[z1 = (z1 \;//\; m_{1,n\to 1}) \;/.\; b\_ \to b,\ \{n, 2, 16\}];$$
$$\{\text{CF@}z1,\ \text{KnotData}[\{8,17\}, \text{"AlexanderPolynomial"}][t]\}$$
$$\left\{-\frac{e^{3b}\,\mathbb{E}[0]}{1-4e^{b}+8e^{2b}-11e^{3b}+8e^{4b}-4e^{5b}+e^{6b}},\ 11-\frac{1}{t^3}+\frac{4}{t^2}-\frac{8}{t}-8t+4t^2-t^3\right\}$$

Demo Programs for 1-Co. — ωεβ/Demo

The Λόγος

$$\Lambda[k\_] := (1-t_k)\left(\alpha^2\beta^2 + 4\alpha\beta\delta\mu + 2\delta^2\mu^2\right)/2 + 2\mu^2(\alpha\beta+\delta\mu)c_k -$$
$$\beta(2\mu-1)(\alpha\beta+2\delta\mu)u_k + 2\beta\delta\mu^2 c_k u_k - \beta^2\delta(3\mu-1)u_k^2/2 +$$
$$\alpha(\alpha\beta+2\delta\mu)w_k + 2\alpha\delta\mu^2 c_k w_k - 2(t_k-1)\delta^2(\alpha\beta+\delta\mu)u_k w_k +$$
$$2\delta^2\mu^2 c_k u_k w_k - \beta\delta^2(2\mu-1)u_k^2 w_k + \alpha^2\delta(1+\mu)w_k^2/2 +$$
$$\alpha\delta^2 u_k w_k^2 - (t_k-1)\delta^4 u_k^2 w_k^2/2;$$

Differential Polynomials

$$\text{DP}_{x\_\to D_{\alpha\_},\,y\_\to D_{\beta\_}}[P\_][f\_] := \text{(* means } P[\partial_\alpha,\partial_\beta][f] \text{ *)}$$
$$\quad \text{Total}[\text{CoefficientRules}[P, \{x,y\}] \;/.$$
$$\quad\quad (\{m\_, n\_\} \to c\_) :\to c\,\text{D}[f, \{\alpha, m\}, \{\beta, n\}]]$$

Utilities

$$\text{CF}[\mathbb{E}[\omega\_, L\_, Q\_, P\_]] := \text{Expand /@ Together /@}$$
$$\quad \mathbb{E}[\omega \;/.\; b_{l\_} :\to \text{Log}[t_l],\ L,\ Q \;/.\; b_{l\_} :\to \text{Log}[t_l],$$
$$\quad\quad P \;/.\; b_{l\_} :\to \text{Log}[t_l]];$$
$$\mathbb{E} \;/\!:\; \mathbb{E}[\omega1\_, L1\_, Q1\_, P1\_]\,\mathbb{E}[\omega2\_, L2\_, Q2\_, P2\_] :=$$
$$\quad \text{CF@}\mathbb{E}\left[\omega1\,\omega2,\ L1+L2,\ \omega2\,Q1+\omega1\,Q2,\ \omega2^4 P1 + \omega1^4 P2\right];$$

Normal Ordering Operators

$$N_{u_{i\_}c_{j\_}\to k\_}[\mathbb{E}[\omega\_, L\_, Q\_, P\_]] := \text{With}\Big[\{q = e^{-\gamma}\beta u_k + \gamma c_k\},\ \text{CF}\Big[$$
$$\quad \mathbb{E}\Big[\omega,\ \gamma c_k + (L \;/.\; c_j \to 0),\ \omega e^{-\gamma}\beta u_k + (Q \;/.\; u_i \to 0),$$
$$\quad\quad e^{-q}\,\text{DP}_{c_j\to D_\gamma,\,u_i\to D_\beta}[P][e^q]\Big] \;/.\; \{\gamma \to \partial_{c_j}L,\ \beta \to \omega^{-1}\partial_{u_i}Q\}\Big]\Big];$$
$$N_{w_{i\_}c_{j\_}\to k\_}[\mathbb{E}[\omega\_, L\_, Q\_, P\_]] := \text{With}\Big[\{q = e^{\gamma}\alpha w_k + \gamma c_k\},\ \text{CF}\Big[$$
$$\quad \mathbb{E}\Big[\omega,\ \gamma c_k + (L \;/.\; c_j \to 0),\ \omega e^{\gamma}\alpha w_k + (Q \;/.\; w_i \to 0),$$
$$\quad\quad e^{-q}\,\text{DP}_{c_j\to D_\gamma,\,w_i\to D_\alpha}[P][e^q]\Big] \;/.\; \{\gamma \to \partial_{c_j}L,\ \alpha \to \omega^{-1}\partial_{w_i}Q\}\Big]\Big];$$
$$N_{w_{i\_}u_{j\_}\to k\_}[\mathbb{E}[\omega\_, L\_, Q\_, P\_]] :=$$
$$\quad \text{With}\Big[\left\{q = (1-t_k)\mu^{-1}\alpha\beta + \mu^{-1}\beta u_k + \mu^{-1}\delta u_k w_k + \mu^{-1}\alpha w_k\right\},\ \text{CF}\Big[$$
$$\quad\quad \mathbb{E}\Big[\mu\omega,\ L,\ \mu\omega q + \mu(Q \;/.\; w_i \,|\, u_j \to 0),$$
$$\quad\quad\quad \mu^4 e^{-q}\,\text{DP}_{w_i\to D_\alpha,\,u_j\to D_\beta}[P][e^q] + \omega^4\Lambda[k]\Big] \;/.$$
$$\quad\quad \mu \to 1 + (t_k-1)\delta \;/.$$
$$\quad\quad \left\{\alpha \to \omega^{-1}\left(\partial_{w_i}Q \;/.\; u_j \to 0\right),\ \beta \to \omega^{-1}\left(\partial_{u_j}Q \;/.\; w_i \to 0\right),\right.$$
$$\quad\quad \left.\delta \to \omega^{-1}\partial_{w_i,u_j}Q\right\}\Big]\Big];$$

Stitching

$$m_{i\_,j\_\to k\_}[Z\_] := \text{Module}\big[\{x, y, z\},$$
$$\quad Z \;//\; N_{w_i c_j\to x} \;//\; N_{w_x u_j\to y} \;//\; \text{ReplaceAll}[\{c_{x|y} \to c_x,\ w_j \to w_y\}] \;//$$
$$\quad\quad N_{u_i c_x\to x} \;//\; \text{ReplaceAll}[z\_{}_{i|j|x|y} \to z_k] \;//\; \text{CF}\big]$$

The Generators

$$R^+_{i\_,j\_} := \mathbb{E}\Big[1,\ b_i c_j,\ u_i w_j,\ -c_i (t_i-1)^2/2 - c_i^2 (t_i-1)^2/2 + c_i c_j \left(t_j^2 - t_i - 2\right)/2 - c_j u_i w_i/2 + c_i (1-t_i) u_i w_i - u_i^2 w_i^2/2 + u_i w_j + c_j t_i u_i w_j/2 + c_i (t_i-2) t_i u_i w_j + c_i (1+t_j) u_j w_j/2 + (t_i-1) u_i^2 w_i w_j - (t_i-2) t_i u_i^2 w_j^2/2\Big];$$

$$R^-_{i\_,j\_} := \mathbb{E}\Big[1,\ -b_i c_j,\ -t_i^{-1} u_i w_j,\ c_i (t_i-1)^2/2 + c_i^2 (t_i-1)^2/2 + c_i c_j \left(2 + t_i - t_j^2\right)/2 + c_j u_i w_i/2 + c_i (t_i-1) u_i w_i + u_i^2 w_i^2/2 + \left(1 - t_i^{-1}\right) u_i w_j/2 + c_i \left(2 t_i - 5 + 3 t_i^{-1}\right) u_i w_j/2 + c_j \left(t_i^{-1} + 1 - t_i^{-1} t_j^2\right) u_i w_j/2 - c_i (t_j+1) u_j w_j/2 + \left(2 - 3 t_i^{-1}\right) u_i^2 w_i w_j/2 + \left(1 + 2 t_i^{-2} - 3 t_i^{-1}\right) u_i^2 w_j^2/2 - t_i^{-1} (1+t_j) u_i u_j w_j^2/2\Big];$$

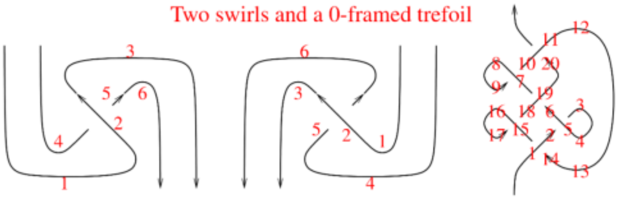
$$ur_{i\_} := \mathbb{E}\left[t_i^{-1/4}, 0, 0, c_i t_i/4 + u_i w_i/8\right];$$

$$nr_{i\_} := \mathbb{E}\left[t_i^{1/4}, 0, 0, -c_i t_i^3/4 - t_i^2 u_i w_i/8\right];$$

$$ul_{i\_} := \mathbb{E}\left[t_i^{1/4}, 0, 0, c_i t_i (4+t_i)/4 - t_i^2 u_i w_i/8\right];$$

$$nl_{i\_} := \mathbb{E}\left[t_i^{-1/4}, 0, 0, -c_i \left(1 + 4 t_i^{-1}\right)/4 + u_i w_i/8\right];$$

Two swirls and a 0-framed trefoil



$$t2 = ur_1\, R^-_{2,5}\, nr_3\, ur_4\, nr_6\ //\ m_{1,2\to 1}\ //\ m_{1,3\to 1}\ //\ m_{4,5\to 4}\ //\ m_{4,6\to 4}$$

$$\mathbb{E}\Big[1,\ -b_1 c_4,\ -\frac{u_1 w_4}{t_1},\ \frac{c_1}{2} + \frac{c_1^2}{2} + c_1 c_4 - c_1 t_1 - c_1^2 t_1 + \frac{1}{2} c_1 c_4 t_1 + \frac{1}{2} c_1 t_1^2 + \frac{1}{2} c_1^2 t_1^2 - \frac{1}{2} c_1 c_4 t_4^2 - c_1 u_1 w_1 + \frac{1}{2} c_4 u_1 w_1 + c_1 t_1 u_1 w_1 + \frac{1}{2} u_1^2 w_1^2 + \frac{3 u_1 w_4}{8} - \frac{5}{2} c_1 u_1 w_4 + \frac{1}{2} c_4 u_1 w_4 - \frac{u_1 w_4}{2 t_1} + \frac{3 c_1 u_1 w_4}{2 t_1} + \frac{c_4 u_1 w_4}{2 t_1} - \frac{1}{8} t_1 u_1 w_4 + c_1 t_1 u_1 w_4 + \frac{t_4 u_1 w_4}{8 t_1} + \frac{t_4^2 u_1 w_4}{8 t_1} - \frac{c_4 t_4^2 u_1 w_4}{2 t_1} - \frac{1}{2} c_1 u_4 w_4 - \frac{1}{2} c_1 t_4 u_4 w_4 + u_1^2 w_1 w_4 - \frac{3 u_1^2 w_1 w_4}{2 t_1} + \frac{1}{2} u_1^2 w_4^2 + \frac{u_1^2 w_4^2}{t_1^2} - \frac{3 u_1^2 w_4^2}{2 t_1} - \frac{u_1 u_4 w_4^2}{2 t_1} - \frac{t_4 u_1 u_4 w_4^2}{2 t_1}\Big]$$

$$t2 == \left(ul_1\, R^-_{2,5}\, nl_3\, ul_4\, nl_6\ //\ m_{1,2\to 1}\ //\ m_{1,3\to 1}\ //\ m_{4,5\to 4}\ //\ m_{4,6\to 4}\right)$$

True

$$z2 = R^+_{1,14}\, R^-_{5,2}\, nr_3\, ul_4\, R^+_{19,6}\, R^-_{7,10}\, nl_8\, ur_9\, R^+_{11,20}\, nr_{12}\, ul_{13}\, R^-_{15,18}\, nl_{16}\, ur_{17};$$

$$(\text{Do}[z2 = z2\ //\ m_{1,k\to 1},\ \{k, 2, 20\}];\ z2 = z2\ /.\ a\_1 :\!\to a)$$

$$\mathbb{E}\Big[-1 + \frac{1}{t} + t,\ 0,\ 0,\ -16 + \frac{9 c}{2} - \frac{2 c}{t^4} + \frac{1}{t^3} + \frac{11 c}{2 t^3} - \frac{4}{t^2} - \frac{8 c}{t^2} + \frac{10}{t} + \frac{4 c}{t} + 18 t - 10 c t - 14 t^2 + 8 c t^2 + 7 t^3 - \frac{3 c t^3}{2} - 2 t^4 - 2 c t^4 + 2 c t^5 - \frac{c t^6}{2} - 4 u w + \frac{2 u w}{t^4} - \frac{7 u w}{2 t^3} + \frac{9 u w}{2 t^2} + \frac{u w}{2 t} + 6 t u w - 2 t^2 u w - \frac{1}{2} t^3 u w + \frac{3}{2} t^4 u w - \frac{1}{2} t^5 u w\Big]$$

```
FromCoefficientRules[
 CoefficientRules[z2[[4]], {c, u, w}] /.
  {(e_ → a_) :→ (e → Simplify[a])}, {c, u, w}]
```

$$-\frac{\left(1-t+t^2\right)^2 \left(-1+2 t-3 t^2+2 t^3\right)}{t^3} - \frac{c \left(1-t+t^2\right)^3 \left(4+t-5 t^2-t^3+t^4\right)}{2 t^4} - \frac{\left(1-t+t^2\right)^3 \left(-4-5 t+t^3\right) u w}{2 t^4}$$

"God created the knots; all else in topology is the work of mortals."

Leopold Kronecker (modified)

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Questions and To Do List. • Clean up and write up. • Implement well, compute for everything in sight. • "Extract the essence" of P. • Why are our quantities polynomials rather than just rational functions? • Bounds on their degrees? • Their integrality ($\mathbb{Z}$) properties? • Can everything be re-stated using integrals ($\int$)? • Find the 2-variable version (for knots). How complex is it? • What about links / closed components? • Fully digest the "expansion" theorem; include cuaps. • Explore the (non-)dependence on R. • Is there a canonical R? • What does "group like" mean? • Strand removal? Strand doubling? Strand reversal? • Say something about knot genus. • Find the EK/AT/KV "vertex". • Use as a playground to study associators/braidors. • Restate in topological language. • Study the associated (v-)braid representations. • Study mirror images and the $\mathfrak{b}^+ \leftrightarrow \mathfrak{b}^-$ involution. • Study ribbon knots. • Make precise the relationship with Γ-calculus and Alexander. • Relate to the coloured Jones polynomial. • Relate with "ordinary" q-algebra. • k-smidgen sl_n, etc. • Are there "solvable" CYBE algebras not arising from semi-simple algebras? • Categorify and appease the Gods.

References.

[Al] J. W. Alexander, *Topological invariants of knots and link,* Trans. Amer. Math. Soc. **30** (1928) 275–306.

[BN1] D. Bar-Natan, *Balloons and Hoops and their Universal Finite Type Invariant, BF Theory, and an Ultimate Alexander Invariant,* ωεβ/KBH, arXiv:1308.1721.

[BN2] D. Bar-Natan, *Polynomial Time Knot Polynomial,* research proposal for the 2017 Killam Fellowship, ωεβ/K17.

[BND] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects I, II, IV,* ωεβ/WKO1, ωεβ/WKO2, ωεβ/WKO4, arXiv:1405.1956, arXiv:1405.1955, arXiv:1511.05624.

[BNG] D. Bar-Natan and S. Garoufalidis, *On the Melvin-Morton-Rozansky conjecture,* Invent. Math. **125** (1996) 103–133.

[BNS] D. Bar-Natan and S. Selmani, *Meta-Monoids, Meta-Bicrossed Products, and the Alexander Polynomial,* J. of Knot Theory and its Ramifications **22-10** (2013), arXiv:1302.5689.

[En] B. Enriquez, *A Cohomological Construction of Quantization Functors of Lie Bialgebras,* Adv. in Math. **197-2** (2005) 430-479, arXiv:math/0212325.

[EK] P. Etingof and D. Kazhdan, *Quantization of Lie Bialgebras, I,* Selecta Mathematica **2** (1996) 1–41, arXiv:q-alg/9506005.

[GST] R. E. Gompf, M. Scharlemann, and A. Thompson, *Fibered Knots and Potential Counterexamples to the Property 2R and Slice-Ribbon Conjectures,* Geom. and Top. **14** (2010) 2305–2347, arXiv:1103.1601.

[GPV] M. Goussarov, M. Polyak, and O. Viro, *Finite type invariants of classical and virtual knots,* Topology **39** (2000) 1045–1068, arXiv:math.GT/9810073.

[Ha] A. Haviv, *Towards a diagrammatic analogue of the Reshetikhin-Turaev link invariants,* Hebrew University PhD thesis, Sep. 2002, arXiv:math.QA/0211031.

[MM] P. M. Melvin and H. R. Morton, *The coloured Jones function,* Commun. Math. Phys. **169** (1995) 501–520.

[PV] M. Polyak and O. Viro, *Gauss Diagram Formulas for Vassiliev Invariants,* Inter. Math. Res. Notices **11** (1994) 445–453.

[Ro] L. Rozansky, *A contribution of the trivial flat connection to the Jones polynomial and Witten's invariant of 3d manifolds, I,* Comm. Math. Phys. **175-2** (1996) 275–296, arXiv:hep-th/9401061.

[Se] P. Ševera, *Quantization of Lie Bialgebras Revisited,* Sel. Math., NS, to appear, arXiv:1401.6164.

Disclaimer. This is all quite new. The overall picture is correct, yet some details might be somewhat off. Many pieces are certainly not in their final form yet. **Help Needed!**

Add a table w/: n_k
pict
a_+
e_+

A AlternatingQ ¿
G genus
U unknotting #
R ribbonQ
C chiralQ