UNC-1610 handout on Oct 2

October 2, 2016 9:39 AM

Dror Bar-Natan: Talks: UNC-1610: ωεβ:=http://drorbn.net/UNC-1610/

Work in Progress! **A Poly-Time Knot Polynomial Via Solvable Approximation**

Abstract. I will construct the first poly-time-computable knot polynomial since Alexander's [Al, 1928] by using some new commutator-calculus techniques and a Lie algebra $\mathfrak{g}_1$ which is at the same time solvable and an approximation of the simple Lie algebra sl_2.

Expected! Finite-type invariants include all coefficients of all quantum knot polynomials (appropriately parametrized), and each is computable in poly-time. Yet

a green paradise

d	2	3	4	5	6	7	8	…
known f.t. invts in $O(n^d)$	1	1	∞	3	4	8	11	…

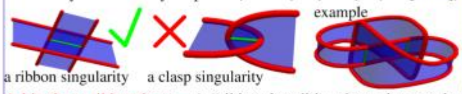
This is an unreasonable picture! So there ought to be further poly-time polynomial invariants.

Also. • The line above the Alexander line in the Melvin-Morton [MM, Ro] expansion of the coloured Jones polynomial. • The 2-loop contribution to the Kontsevich integral.


Rozansky

Paradise! Foremost reason: *OBVIOUSLY.* Cf. proving (incomputable A)=(incomputable B), or categorifying (incomputable C).

Secondary reason: may disprove {ribbon} = {slice}: (see [BN2])



a ribbon singularity   a clasp singularity

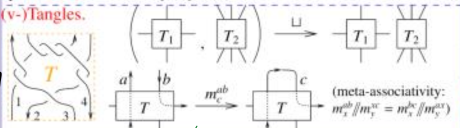
A bit about ribbon knots. A "ribbon knot" is a knot that can be presented as the boundary of a disk that has "ribbon singularities", but no "clasp singularities". A "slice knot" is a knot in $S^3 = \partial B^4$ which is the boundary of a non-singular disk in B^4. Every ribbon knots is clearly slice, yet,

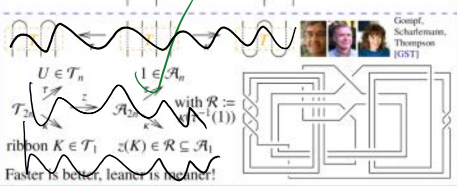
Conjecture. Some slice knots are not ribbon.

A

~~**Fox-Milnor.** The Alexander polynomial of a ribbon knot is always of the form $A(t) = f(t)f(1/t)$. (also for slice)~~

(v-)Tangles.





The Gold Standard is set by the "Γ-calculus" Alexander formulas [BNS, BN1]. An S-component tangle T has $\Gamma(T) \in R_S \times M_{S\times S}(R_S) = \left\{\begin{array}{c|c} \omega & S \\ \hline S & A\end{array}\right\}$ with $R_S := \mathbb{Z}(\{t_a : a \in S\})$:

$$\left({}_a\!\nearrow\!\!\!\!\nwarrow_b,\ {}_b\!\nwarrow\!\!\!\!\nearrow_a\right) \to \begin{array}{c|cc} 1 & a & b \\ \hline a & 1 & 1-t_a^{\pm 1} \\ b & 0 & t_a^{\pm 1}\end{array} \qquad T_1 \sqcup T_2 \to \begin{array}{c|cc} \omega_1\omega_2 & S_1 & S_2 \\ \hline S_1 & A_1 & 0 \\ S_2 & 0 & A_2\end{array}$$

$$\begin{array}{c|ccc} \omega & a & b & S \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi\end{array} \xrightarrow[t_a, t_b \to t_c]{m^{ab}_c} \left(\begin{array}{c|cc} (1-\beta)\omega & c & S \\ \hline c & \gamma + \frac{\alpha\delta}{1-\beta} & \epsilon + \frac{\delta\theta}{1-\beta} \\ S & \phi + \frac{\alpha\psi}{1-\beta} & \Xi + \frac{\psi\theta}{1-\beta}\end{array}\right)$$

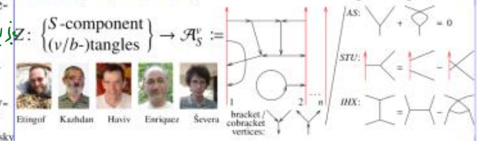
For long knots, ω is Alexander, and that's the fastest Alexander algorithm I know!

Dunfield: 1000-crossing fast.

refer the o-smidgen discussion.

Theorem [EK, Ha, En, Se]. There is a "homomorphic expansion"

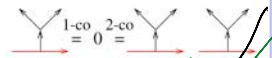
$$Z: \left\{\begin{array}{l} S\text{-component} \\ (v/b\text{-})\text{tangles}\end{array}\right\} \to \mathcal{A}^v_S :=$$

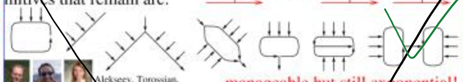


(it is enough to know Z on ⤫ and have disjoint union and stitching formulas) …exponential and too hard!

Idea. Look for "ideal" quotients of $\mathcal{A}^v_S$ that have poly-sized descriptions; …specifically, limit the co-brackets.

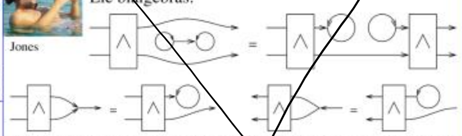
1-co and 2-co, aka TC and TC^2, on the right. The primitives that remain are:





…manageable but still exponential!

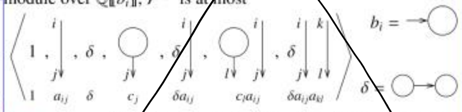
The 2D relations come from the relation with 2D Lie bialgebras:



We let $\mathcal{A}^{2,2}$ be $\mathcal{A}^v$ modulo 2-co and 2D, and $z^{2,2}$ be the projection of $\log Z$ to $\mathcal{P}^{2,2} := \pi\mathcal{P}^v$, where $\mathcal{P}^v$ are the primitives of $\mathcal{A}^v$.

Main Claim. $z^{2,2}$ is poly-time computable.

Main Point. $\mathcal{P}^{2,2}$ is poly-size, so how hard can it be? Indeed, as a module over $\mathbb{Q}[\![b_i]\!]$, $\mathcal{P}^{2,2}$ is at most



Claim. $R_{jk} = e^{a_{jk}} e^{\rho_{jk}}$ is a solution of the Yang-Baxter / R3 equation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ in $\exp\mathcal{P}^{2,2}$, with $\rho_{jk} :=$

$$\psi(b_j)\left(-c_k + \frac{c_k a_{jk}}{b_j} - \frac{\delta a_{jk} a_{jk}}{b_j^2}\right) + \frac{\phi(b_j)\psi(b_k)}{b_k\phi(b_k)}\left(c_k a_{kk} - \frac{\delta a_{jk} a_{kk}}{b_j}\right),$$

and with $\phi(x) := e^{-x} - 1 = -x + x^2/2 - \ldots$, and $\psi(x) = ((x+2)e^{-x} - 2 + x)/(2x) = x^2/12 - x^3/24 + \ldots$.

But how do we multiply in $\exp(\mathcal{P}^{2,2})$? How do we stitch?

Videos of a 4-hour version of this talk are at ωεβ/LD. **Videos** of private seminar meetings are at ωεβ/PP.

Many thanks: Vo, Halacheva, Dalvit, Ens, Lee (van der Veen, Schaveling)

A: in [Killam] I list five criteria that an invariant needs to meet to have a fair chance of detecting non-ribbons. ours meets all 5.

1-Smidgen sl_2 (with van der Veen). Let $\mathfrak{g}_1$ be the 4D Lie algebra $\mathfrak{g}_1 = \langle b, c, u, w\rangle$ over $\mathbb{Q}[\epsilon]/(\epsilon^2 = 0)$, with b central and $[w, c] = w$, $[c, u] = u$, and $[u, w] = b - 2\epsilon c$, with CYBE $r_{ij} = (b_i - \epsilon c_i)c_j + u_i w_j$ in $\mathcal{U}(\mathfrak{g}_1)^{\otimes\{i,j\}}$. Over $\mathbb{Q}$, $\mathfrak{g}_1$ is a solvable approximation of sl_2: $\mathfrak{g}_1 \supset \langle b, u, w, \epsilon b, \epsilon c, \epsilon u, \epsilon w\rangle \supset \langle b, \epsilon b, \epsilon c, \epsilon u, \epsilon w\rangle \supset 0$. In a certain sense, $\mathfrak{g}_1$ is more valuable than sl_2. (note: $\deg(b, c, u, w, \epsilon) = (1, 0, 1, 0, 1)$)

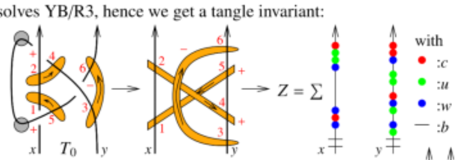
0-Smidgen sl_2 ☺. Let $\mathfrak{g}_0$ be $\mathfrak{g}_1$ at $\epsilon = 0$, or $\mathbb{Q}\langle b, c, u, w\rangle/([b, \cdot] = 0, [c, u] = u, [c, w] = -w, [u, w] = b$ with $r_{ij} = b_i c_j + u_i w_j$. It is $\mathfrak{a}^* \rtimes \mathfrak{a}$ where $\mathfrak{a}$ is the 2D Lie algebra $\mathbb{Q}\langle b, u\rangle$ and (c, w) is the dual basis of (b, u). It is even more valuable than $\mathfrak{g}_1$, but topology already got by other means almost everything $\mathfrak{g}_0$ has to give.

How did these arise? $sl_2 = \mathfrak{b}^+ \oplus \mathfrak{b}^-/\mathfrak{h} =: sl_2^+/\mathfrak{h}$, where $\mathfrak{b}^+ = \langle c, w\rangle/[w, c] = w$ is a Lie bialgebra with $\delta\colon \mathfrak{b}^+ \to \mathfrak{b}^+ \otimes \mathfrak{b}^+$ by $\delta\colon (c, w) \mapsto (0, c \wedge w)$. Going back, $sl_2^+ = \mathcal{D}(\mathfrak{b}^+) = (\mathfrak{b}^+)^* \oplus \mathfrak{b}^+ = \langle b, u, c, w\rangle/\cdots$. **Idea.** Replace $\delta \to \epsilon\delta$ over $\mathbb{Q}[\epsilon]/(\epsilon^{k+1} = 0)$. At $k = 0$, get $\mathfrak{g}_0$. At $k = 1$, get $[w, c] = w$, $[w, b'] = -\epsilon w$, $[c, u] = u$, $[b', u] = -\epsilon u$, $[b', c] = 0$, and $[u, w] = b' - \epsilon c$. Now note that $b' + \epsilon c$ is central, so switch to $b := b' + \epsilon c$. This is $\mathfrak{g}_1$.

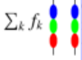
0-Smidgen Invariants. $r = Id \in \mathfrak{b}^- \otimes \mathfrak{b}^+$ solves the CYBE $[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$ in $\mathcal{U}(\mathfrak{g}_0)^{\otimes 3}$ and, by luck,

$= R_{ij} = e^{r_{ij}} = e^{b_i c_j + u_i w_j} \in \mathcal{U}(\mathfrak{g}_{0,i} \oplus \mathfrak{g}_{0,j})$

solves YB/R3, hence we get a tangle invariant:



Goal. Sort Z to be as on the right, with $f_k \in \mathbb{Q}[\![b_i]\!]$. Better, with $\zeta \in \mathbb{Q}[\![b_x, c_x, u_x, w_x, b_y, c_y, u_y, w_y]\!]$, write



$$Z = \mathbb{O}\left(\zeta | x\colon c_x u_x w_x,\ y\colon c_y u_y w_y\right) \qquad \text{(cuw form)}$$

Here $\mathbb{O}(poly \mid specs)$ plants the variables of *poly* in $\mathcal{S}(\oplus_i \mathfrak{g})$ on several tensor copies of $\mathcal{U}(\mathfrak{g})$ according to *specs*. E.g.,

$$\mathbb{O}\left(c_1^3 u_1 c_2 e^{u_3} w_3^9 | x\colon w_3 c_1,\ y\colon u_1 u_3 c_2\right) = w^9 c^3 \otimes u e^u c \in \mathcal{U}(\mathfrak{g})_x \otimes \mathcal{U}(\mathfrak{g})_y$$

Lemma. $R_{ij} = e^{b_i c_j + u_i w_j} = \mathbb{O}\left(\exp\left(b_i c_j + \frac{e^{b_i}-1}{b_i} u_i w_j\right) | i\colon u_i,\ j\colon c_j w_j\right)$

$= \sum_{m,n} \frac{b_i^{m-n}(e^{b_i}-1)^n}{m!n!} u^n \otimes c^m w^n$.

Example. $Z(T_0) =$

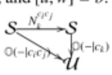
$$\mathbb{O}\Big(1 \exp\left(b_5 c_1 + \tfrac{e^{b_5}-1}{b_5} u_5 w_1 + b_2 c_4 + \tfrac{e^{b_2}-1}{b_2} u_2 w_4 - b_3 c_6 + \tfrac{e^{-b_3}-1}{b_3} u_3 w_6\right) \Big|\ x\colon c_1 w_1 u_2,\ y\colon u_3 c_4 w_4 u_5 c_6 w_6\Big)$$

$$= \mathbb{O}\left(? | x\colon c_x u_x w_x,\ y\colon c_y u_y w_y\right)$$

$\mathbb{O}(\omega e^{L+Q})$: L bilinear in b_i and c_i, and Q a balanced quadratic in u_i and w_i with coefficients in $\mathbb{Q}(b_i, e^{b_i}) \ni \omega$. "Admissible"

The Big $\mathfrak{g}_0$ Lemma. Under $[c, u] = u$, $[c, w] = -w$, and $[u, w] = b$:

1. $N_k^{c_i c_j} := \mathbb{O}(\zeta | c_i c_j) \stackrel{\to}{=} \mathbb{O}(\zeta/(c_i, c_j \to c_k) | c_k)$ (Meaning, $N_k^{c_i c_j}\colon \zeta \mapsto (\zeta/(c_i, c_j \to c_k))$ and the diagram commutes. Trivial, also for b, u, w.)



2a. $N^{uc} := \mathbb{O}(e^{\gamma c + \beta u} | uc) \stackrel{\to}{=} \mathbb{O}(e^{\gamma c + e^{-\gamma}\beta u} | cu)$ (means $e^{\beta u} e^{\gamma c} = e^{\gamma c} e^{e^{-\gamma}\beta u}$

2b. $N^{wc} := \mathbb{O}(e^{\gamma c + \alpha w} | wc) \stackrel{\to}{=} \mathbb{O}(e^{\gamma c + e^{\gamma}\alpha w} | cw)$ …in the $\{ax + b\}$ group)

3. $\mathbb{O}(e^{\alpha w + \beta u} | wu) = \mathbb{O}(e^{-b\alpha\beta + \alpha w + \beta u} | uw)$ (the Weyl relations)

4. $\mathbb{O}(e^{\delta uw} | wu) e^{\beta u} = e^{\nu\beta u} \mathbb{O}(e^{\delta uw} | wu)$, with $\nu = (1 + b\delta)^{-1}$

(a. expand and crunch. b. use $w = b\hat{x}$, $u = \partial_x$. c. use "scatter and glow".)

5. $\mathbb{O}(e^{\delta uw} | wu) = \mathbb{O}(\nu e^{\nu\delta uw} | uw)$ (same techniques)

6. $N^{wu} := \mathbb{O}(e^{\beta u + \alpha w + \delta uw} | wu) \stackrel{\to}{=} \mathbb{O}(\nu e^{-b\nu\alpha\beta + \nu\alpha w + \nu\beta u + \nu\delta uw} | uw)$

Sneaky. α may contain (other) u's, β may contain (other) w's.

Strand Stitching, m_k^{ij}, is defined as the composition

$$c_i u_i \overline{w_i c_j} u_j w_j \xrightarrow{N_k^{w_i c_j}} c_i \overline{u_i c_k}\, \overline{w_k u_j} w_j \xrightarrow{N_k^{u_i c_k} /\!/ N_k^{w_k u_j}} \overline{c_i c_k}\, \overline{u_k u_k}\, \overline{w_k w_j} \xrightarrow{N_k^{c_i c_k} /\!/ - - /\!/ N_k^{w_k w_j}} c_k u_k w_k$$

1-Smidgen Invariants. Much is the same:

The Big $\mathfrak{g}_1$ Lemma. Parts 1 and 2 are the same, yet

6. $\mathbb{O}\left(e^{\alpha w + \beta u + \delta uw} | wu\right) = \mathbb{O}\left(\nu(1 + \epsilon\nu\Lambda) e^{\nu(-b\alpha\beta + \alpha w + \beta u + \delta uw)} | cuw\right)$

Here Λ is for Λόγος, "a principle of order and knowledge", a balanced quartic in α, β, c, u, and w:

$$\begin{aligned}\Lambda = &- b\nu\left(\nu^2\alpha^2\beta^2 + 4\delta\nu\alpha\beta + 2\delta^2\right)/2 - \delta\nu^3(3b\delta + 2)\beta^2 u^2/2 \\ &- b\delta^4\nu^3 u^2 w^2/2 - \delta^2\nu^3(2b\delta + 1)\beta u^2 w \\ &- \nu^2(2b\delta + 1)(\nu\alpha\beta + 2\delta)\beta u - 2b\delta^2\nu^2(\nu\alpha\beta + \delta)uw \\ &+ \delta\nu^3(b\delta + 2)\alpha^2 w^2/2 + 2(\nu\alpha\beta + \delta)c + 2\delta\nu\beta cu + 2\delta^2\nu cuw \\ &+ 2\delta\nu\alpha cw + \delta^2\nu^3\alpha u w^2 + \nu^2(\nu\alpha\beta + 2\delta)\alpha w.\end{aligned}$$

Proof. A brutal hell.

Problem. We now need to normal-order perturbed Gaussians!

Solution. Borrow some tactics from QFT:

$$\mathbb{O}(\epsilon P(c, u) e^{\gamma c + \beta u} | uc) = \mathbb{O}(\epsilon P(\partial_\gamma, \partial_\beta) e^{\gamma c + \beta u} | uc) = \mathbb{O}(\epsilon P(\partial_\gamma, \partial_\beta) e^{\gamma c + e^{-\gamma}\beta u} | cu),$$

and likewise

$$\mathbb{O}\left(\epsilon P(u, w) e^{\alpha w + \beta u + \delta uw} | wu\right) = \mathbb{O}\left(\epsilon P(\partial_\beta, \partial_\alpha) \nu e^{\nu(-b\alpha\beta + \alpha w + \beta u + \delta uw)} | cuw\right)$$

Note. Strand stitching requires a tiny extra step.

Finally, the values of the generators ⤫, ⤬, $\overrightarrow{n}$, $\overleftarrow{n}$, $\underset{\to}{u}$, and $\underset{\leftarrow}{u}$, are set by brutally solving many equations, non-uniquely.

Pragmatic Simplifications. Get rid of $\zeta = (e^b - 1)/b$ factors by rescaling $u \to \bar{u} = \zeta u$. Complement this with $\beta \to \bar{\beta} = \zeta^{-1}\beta$, $\delta \to \bar{\delta} = \zeta^{-1}\delta$, $\epsilon \to \bar{\epsilon} = \zeta^{-1}\epsilon$. Simplify further by naming $e^b \to t$; e.g., $\nu \to \bar{\nu} = (1 + (t - 1)\delta)^{-1}$. Get confused by renaming $(\bar{u}, \bar{\beta}, \bar{\delta}, \bar{\nu}) \to (u, \beta, \delta, \nu)$, and more confused by working with $\mu = \nu^{-1}$ and $\mathbb{E}(\omega, L, Q, P) := \omega^{-1}(1 + \epsilon\omega^{-4}P)e^{L + \omega^{-1}Q}$, where $\omega \in R := \mathbb{Q}(t_k)$, $L = \sum l_{ij} b_i c_j$ with $l_{ij} \in \mathbb{Z}$, $Q = \sum q_{ij} u_i w_j$ with $q_{ij} \in R$, and P is a balanced quartic polynomial in c_i, u_i, and w_i with coefficients in R. Magically, all coefficients are now Laurent polynomials in the t_k's.

Rough complexity estimate, after $t_k \to t$. n: xing number; w: width, maybe $\sim \sqrt{n}$.

$$\underbrace{n}_{A} \overbrace{\sum_{d=0}^{4} \underbrace{w^{4-d}}_{E} \underbrace{w^d}_{F}}^{B} \underbrace{n^2}_{G} = n^3 w^4 \in [n^5, n^7]$$

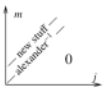
A: go over stitchings in order. B: multiplication ops per $N^{u_i w_j}$. d: deg of u_i, w_j in P. E: #terms of deg d in P. F: ops per term. G: cost per polynomial multiplication op.

Expectation. Our invariant is the "1-higher diagonal" in the MMR expansion of the coloured Jones polynomial J_λ.

Theorem ([BNG], conjectured [MM], elucidated [Ro]). Let $J_d(K)$ be the coloured Jones polynomial of K, in the d-dimensional representation of $sl(2)$. Writing

$$\left.\frac{(q^{1/2} - q^{-1/2})J_d(K)}{q^{d/2} - q^{-d/2}}\right|_{q=e^\hbar} = \sum_{j,m\geq 0} a_{jm}(K) d^j \hbar^m,$$

"below diagonal" coefficients vanish, $a_{jm}(K) = 0$ if $j > m$, and "on diagonal" coefficients give the inverse of the Alexander polynomial: $\left(\sum_{m=0}^{\infty} a_{mm}(K)\hbar^m\right) \cdot A(K)(e^\hbar) = 1$.



Rearrange as:

1. general discussion for any $A/\mathcal{U}(\mathfrak{g})^{\otimes S}$-valued invariant.
2. Representation theory, success & failure
3. Idea: Work in a space of "formulas of a specific type" for elements of $\mathcal{U}(\mathfrak{g})^{\otimes S}$
4. The $\mathfrak{g}_0$ discussion
5. How do $\mathfrak{g}_0$ & $\mathfrak{g}_1$ arise?
6. The $\mathfrak{g}_1$ discussion.

Demo Programs for 0-Co.

ωεβ/Demo

The *R*-matrices

```
R⁺θ,i_,j_ := 𝔼[b_i c_j + b_i^-1 (e^(b_i) - 1) u_i w_j];
R⁻θ,i_,j_ := 𝔼[-b_i c_j + b_i^-1 (e^(-b_i) - 1) u_i w_j];
```

Utilities

```
CF[ω_. 𝔼[Q_]] := Simplify[ω] 𝔼[Simplify[Q]];
𝔼 /: 𝔼[Q1_] 𝔼[Q2_] := CF@𝔼[Q1 + Q2];
ω1_. 𝔼[Q1_] ≡ ω2_. 𝔼[Q2_] := Simplify[ω1 == ω2 ∧ Q1 == Q2];
```

Normal Ordering Operators

```
N_{u_i_ c_j_ → k_}[ω_. 𝔼[Q_]] := CF[
   ω 𝔼[e^-γ β u_k + γ c_k + (Q /. c_j | u_i → 0)] /. {γ → ∂_{c_j} Q, β → ∂_{u_i} Q}];
N_{w_i_ c_j_ → k_}[ω_. 𝔼[Q_]] := CF[
   ω 𝔼[e^γ α w_k + γ c_k + (Q /. c_j | w_i → 0)] /. {γ → ∂_{c_j} Q, α → ∂_{w_i} Q}];
N_{w_i_ u_j_ → k_}[ω_. 𝔼[Q_]] := CF[
   ν ω 𝔼[-b_k ν α β + ν β u_k + ν δ u_k w_k + ν α w_k + (Q /. w_i | u_j → 0)] /.
    ν → (1 + b_k δ)^-1 /.
   {α → ∂_{w_i} Q /. u_j → 0, β → ∂_{u_j} Q /. w_i → 0, δ → ∂_{w_i,u_j} Q}];
```

Stitching

```
m_{i_,j_→k_}[ω_. 𝔼[Q_]] := CF[Module[{x},
   (ω 𝔼[Q] /. b_{i|j} → b_k // N_{w_i c_j→x} // N_{u_i c_x→x} // N_{w_x u_j→x}) /.
    {c_i → c_k, w_j → w_k, y_x ⧴ y_k}]]
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Some calculations for T_0

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T_{θ,0} = R⁺θ,5,1 R⁺θ,2,4 R⁻θ,3,6
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$$\mathbb{E}\left[b_5 c_1 + b_2 c_4 - b_3 c_6 + \frac{(-1+e^{b_5}) u_5 w_1}{b_5} + \frac{(-1+e^{b_2}) u_2 w_4}{b_2} + \frac{(-1+e^{-b_3}) u_3 w_6}{b_3}\right]$$

```
T_{θ,1} = T_{θ,0} // N_{u3 c4→4}
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$$\mathbb{E}\left[b_5 c_1 + b_2 c_4 - b_3 c_6 + \frac{(-1+e^{b_5}) u_5 w_1}{b_5} + \frac{(-1+e^{b_2}) u_2 w_4}{b_2} + \frac{e^{-b_2}(-1+e^{-b_3}) u_4 w_6}{b_3}\right]$$

```
T_{θ,2} = T_{θ,1} // N_{w4 u5→4}
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$$\mathbb{E}\left[b_5 c_1 + b_2 c_4 + \frac{(-1+e^{b_5})(-(-1+e^{b_2}) b_4 u_2 + b_2 u_4) w_1}{b_2 b_5} + \frac{(-1+e^{b_2}) u_2 w_4}{b_2} - \frac{b_3^2 c_6 + e^{-b_2-b_3}(-1+e^{b_3}) u_4 w_6}{b_3}\right]$$

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T_{θ,2} // N_{w1 u2→1}
```

$$\frac{1}{1-\frac{(-1+e^{b_2})(-1+e^{b_5}) b_1 b_4}{b_2 b_5}}\,\mathbb{E}\Big[\frac{1}{b_3((-1+e^{b_2})(-1+e^{b_5}) b_1 b_4 - b_2 b_5)}\big(b_3 b_5 ((-1+e^{b_2})(-1+e^{b_5}) b_1 b_4 - b_2 b_5) c_1 + b_2 b_3 ((-1+e^{b_2})(-1+e^{b_5}) b_1 b_4 - b_2 b_5) c_4 + (-1+e^{b_2})(-1+e^{b_5}) b_3 b_4 u_1 w_1 - (-1+e^{b_5}) b_2 b_3 u_4 w_1 - (-1+e^{b_2}) b_3 b_5 u_1 w_4 + (-1+e^{b_2})(-1+e^{b_5}) b_1 b_3 u_4 w_4 - ((-1+e^{b_2})(-1+e^{b_5}) b_1 b_4 - b_2 b_5)(b_3^2 c_6 + e^{-b_2-b_3}(-1+e^{b_3}) u_4 w_6)\big)\Big]$$

```
T_{θ,0} // m_{1,2→1} // m_{3,4→3} // m_{3,5→3} // m_{3,6→3}
```

$$\frac{1}{1-(-1+e^{b_1})(-1+e^{b_3})}\,\mathbb{E}\Big[b_3 c_1 + b_1 c_3 - b_3 c_3 + \frac{(-1+e^{b_1})(-1+e^{b_3}) u_1 w_1}{(-e^{b_1}-e^{b_3}+e^{b_1+b_3}) b_1} - \frac{e^{-b_3}(-1+e^{b_1})(b_3 u_1 - e^{b_3}(-1+e^{b_3}) b_1 u_3) w_3}{(-e^{b_1}-e^{b_3}+e^{b_1+b_3}) b_1 b_3} + \frac{e^{-b_1}(-1+e^{b_3}) u_3 (-e^{b_1+b_3} w_1 + (e^{b_1}+e^{b_3}-e^{b_1+b_3}) w_3)}{(-e^{b_1}-e^{b_3}+e^{b_1+b_3}) b_3}\Big]$$

Verifying meta-associativity

```
Q0 = 𝔼[Sum[f_i c_i, {i, 3}] + Sum[f_{i,j} u_i w_j, {i, 3}, {j, 3}]]
```

$$\mathbb{E}[c_1 f_1 + c_2 f_2 + c_3 f_3 + u_1 w_1 f_{1,1} + u_1 w_2 f_{1,2} + u_1 w_3 f_{1,3} + u_2 w_1 f_{2,1} + u_2 w_2 f_{2,2} + u_2 w_3 f_{2,3} + u_3 w_1 f_{3,1} + u_3 w_2 f_{3,2} + u_3 w_3 f_{3,3}]$$

```
(Q0 // m_{1,2→1} // m_{1,3→1}) ≡ (Q0 // m_{2,3→2} // m_{1,2→1})
True
```

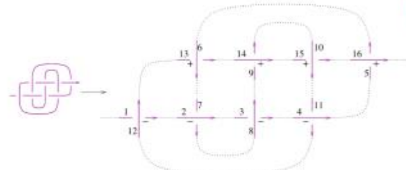
Testing R3

```
t1 = R⁺θ,1,2 R⁺θ,3,4 R⁺θ,5,6 // m_{3,5→x} // m_{1,6→y} // m_{2,4→z}
```

$$\mathbb{E}\left[b_x (c_y + c_z) + \frac{(-1+e^{b_x}) u_x (w_y + w_z)}{b_x} + \frac{b_y^2 c_z + (-1+e^{b_y}) u_y w_z}{b_y}\right]$$

```
t1 ≡ (R⁺θ,1,2 R⁺θ,3,4 R⁺θ,5,6 // m_{1,3→x} // m_{2,5→y} // m_{4,6→z})
True
```

8_{17}




```
z1 = R⁻θ,12,1 R⁻θ,2,7 R⁻θ,8,3 R⁻θ,4,11 R⁺θ,16,5 R⁺θ,6,13 R⁺θ,14,9 R⁺θ,10,15;
Do[z1 = (z1 // m_{1,n→1}) /. b_ → b, {n, 2, 16}];
{CF@z1, KnotData[{8, 17}, "AlexanderPolynomial"][t]}
```

$$\left\{-\frac{e^{3b}\,\mathbb{E}[0]}{1-4e^{b}+8e^{2b}-11e^{3b}+8e^{4b}-4e^{5b}+e^{6b}},\ 11-\frac{1}{t^3}+\frac{4}{t^2}-\frac{8}{t}-8t+4t^2-t^3\right\}$$

Demo Programs for 1-Co.

ωεβ/Demo

The Λόγος

```
Λ[k_] := (1 - t_k) (α² β² + 4 α β δ μ + 2 δ² μ²)/2 + 2 μ² (α β + δ μ) c_k -
   β (2 μ - 1) (α β + 2 δ μ) u_k + 2 β δ μ² c_k u_k - β² δ (3 μ - 1) u_k²/2 +
   α (α β + 2 δ μ) w_k + 2 α δ μ² c_k w_k - 2 (t_k - 1) δ² (α β + δ μ) u_k w_k +
   2 δ² μ² c_k u_k w_k - β δ² (2 μ - 1) u_k² w_k + α² δ (1 + μ) w_k²/2 +
   α δ² u_k w_k² - (t_k - 1) δ⁴ u_k² w_k²/2;
```

Differential Polynomials

```
DP_{x_→D_α_, y_→D_β_}[P_][f_] := (* means P[∂_α,∂_β][f] *)
 Total[CoefficientRules[P, {x, y}] /.
   ({m_, n_} → c_) ⧴ c D[f, {α, m}, {β, n}]]
```

Utilities

```
CF[𝔼[ω_, L_, Q_, P_]] := Expand /@ Together /@
   𝔼[ω /. b_l_ ⧴ Log[t_l], L, Q /. b_l_ ⧴ Log[t_l],
    P /. b_l_ ⧴ Log[t_l]];
𝔼 /: 𝔼[ω1_, L1_, Q1_, P1_] 𝔼[ω2_, L2_, Q2_, P2_] :=
  CF@𝔼[ω1 ω2, L1 + L2, ω2 Q1 + ω1 Q2, ω2⁴ P1 + ω1⁴ P2];
```

Normal Ordering Operators

```
N_{u_i_ c_j_ → k_}[𝔼[ω_, L_, Q_, P_]] := With[{q = e^-γ β u_k + γ c_k}, CF[
   𝔼[ω, γ c_k + (L /. c_j → 0), ω e^-γ β u_k + (Q /. u_i → 0),
     e^-q DP_{c_j→D_γ, u_i→D_β}[P][e^q]] /. {γ → ∂_{c_j} L, β → ω^-1 ∂_{u_i} Q}]];
N_{w_i_ c_j_ → k_}[𝔼[ω_, L_, Q_, P_]] := With[{q = e^γ α w_k + γ c_k}, CF[
   𝔼[ω, γ c_k + (L /. c_j → 0), ω e^γ α w_k + (Q /. w_i → 0),
     e^-q DP_{c_j→D_γ, w_i→D_α}[P][e^q]] /. {γ → ∂_{c_j} L, α → ω^-1 ∂_{w_i} Q}]];
N_{w_i_ u_j_ → k_}[𝔼[ω_, L_, Q_, P_]] :=
  With[{q = (1 - t_k) μ^-1 α β + μ^-1 β u_k + μ^-1 δ u_k w_k + μ^-1 α w_k}, CF[
   𝔼[μ ω, L, μ ω q + μ (Q /. w_i | u_j → 0),
      μ⁴ e^-q DP_{w_i→D_α, u_j→D_β}[P][e^q] + ω⁴ Λ[k]] /.
     μ → 1 + (t_k - 1) δ /.
    {α → ω^-1 (∂_{w_i} Q /. u_j → 0), β → ω^-1 (∂_{u_j} Q /. w_i → 0),
     δ → ω^-1 ∂_{w_i,u_j} Q}]];
```

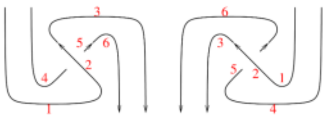
Stitching

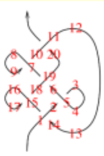
```
m_{i_,j_→k_}[Z_] := Module[{x, y, z},
  Z // N_{w_i c_j→x} // N_{w_x u_j→y} // ReplaceAll[{c_{x|y} → c_x, w_j → w_y}] //
    N_{u_i c_x→x} // ReplaceAll[z_{_i|j|x|y} → z_k] // CF]
```

The Generators

$$
\begin{aligned}
&R^{+}_{i\_,j\_} := \mathbb{E}\Big[1,\ b_i c_j,\ u_i w_j,\\
&\quad -c_i (t_i-1)^2/2 - c_i^2 (t_i-1)^2/2 + c_i c_j \left(t_j^2 - t_i - 2\right)/2 -\\
&\quad c_j u_i w_i/2 + c_i (1-t_i) u_i w_i - u_i^2 w_i^2/2 + u_i w_j + c_j t_i u_i w_j/2 +\\
&\quad c_i (t_i-2) t_i u_i w_j + c_i (1+t_j) u_j w_j/2 + (t_i-1) u_i^2 w_i w_j -\\
&\quad (t_i-2) t_i u_i^2 w_j^2/2\Big];\\
&R^{-}_{i\_,j\_} := \mathbb{E}\Big[1,\ -b_i c_j,\ -t_i^{-1} u_i w_j,\\
&\quad c_i (t_i-1)^2/2 + c_i^2 (t_i-1)^2/2 + c_i c_j \left(2 + t_i - t_j^2\right)/2 +\\
&\quad c_j u_i w_i/2 + c_i (t_i-1) u_i w_i + u_i^2 w_i^2/2 + \left(1 - t_i^{-1}\right) u_i w_j/2 +\\
&\quad c_i \left(2 t_i - 5 + 3 t_i^{-1}\right) u_i w_j/2 + c_j \left(t_i^{-1} + 1 - t_i^{-1} t_j^2\right) u_i w_j/2 -\\
&\quad c_i (t_j+1) u_j w_j/2 + \left(2 - 3 t_i^{-1}\right) u_i^2 w_i w_j/2 +\\
&\quad \left(1 + 2 t_i^{-2} - 3 t_i^{-1}\right) u_i^2 w_j^2/2 - t_i^{-1} (1+t_j) u_i u_j w_j^2/2\Big];\\
&ur_{i\_} := \mathbb{E}\left[t_i^{-1/4},\ 0,\ 0,\ c_i t_i/4 + u_i w_i/8\right];\\
&nr_{i\_} := \mathbb{E}\left[t_i^{1/4},\ 0,\ 0,\ -c_i t_i^3/4 - t_i^2 u_i w_i/8\right];\\
&ul_{i\_} := \mathbb{E}\left[t_i^{1/4},\ 0,\ 0,\ c_i t_i (4+t_i)/4 - t_i^2 u_i w_i/8\right];\\
&nl_{i\_} := \mathbb{E}\left[t_i^{-1/4},\ 0,\ 0,\ -c_i \left(1 + 4 t_i^{-1}\right)/4 + u_i w_i/8\right];
\end{aligned}
$$

Two swirls and a 0-framed trefoil





$$t2 = ur_1\, R^{-}_{2,5}\, nr_3\, ur_4\, nr_6 \ // \ m_{1,2\to1} \ // \ m_{1,3\to1} \ // \ m_{4,5\to4} \ // \ m_{4,6\to4}$$

$$
\begin{aligned}
&\mathbb{E}\Big[1,\ -b_1 c_4,\ -\tfrac{u_1 w_4}{t_1},\\
&\quad \tfrac{c_1}{2} + \tfrac{c_1^2}{2} + c_1 c_4 - c_1 t_1 - c_1^2 t_1 + \tfrac{1}{2} c_1 c_4 t_1 + \tfrac{1}{2} c_1 t_1^2 + \tfrac{1}{2} c_1^2 t_1^2 -\\
&\quad \tfrac{1}{2} c_1 c_4 t_4^2 - c_1 u_1 w_1 + \tfrac{1}{2} c_4 u_1 w_1 + c_1 t_1 u_1 w_1 + \tfrac{1}{2} u_1^2 w_1^2 + \tfrac{3 u_1 w_4}{8} -\\
&\quad \tfrac{5}{2} c_1 u_1 w_4 + \tfrac{1}{2} c_4 u_1 w_4 - \tfrac{u_1 w_4}{2 t_1} + \tfrac{3 c_1 u_1 w_4}{2 t_1} + \tfrac{c_4 u_1 w_4}{2 t_1} - \tfrac{1}{8} t_1 u_1 w_4 +\\
&\quad c_1 t_1 u_1 w_4 + \tfrac{t_4 u_1 w_4}{8 t_1} + \tfrac{t_4^2 u_1 w_4}{8 t_1} - \tfrac{c_4 t_4^2 u_1 w_4}{2 t_1} - \tfrac{1}{2} c_1 u_4 w_4 - \tfrac{1}{2} c_1 t_4 u_4 w_4 +\\
&\quad u_1^2 w_1 w_4 - \tfrac{3 u_1^2 w_1 w_4}{2 t_1} + \tfrac{1}{2} u_1^2 w_4^2 + \tfrac{u_1^2 w_4^2}{t_1^2} - \tfrac{3 u_1^2 w_4^2}{2 t_1} - \tfrac{u_1 u_4 w_4^2}{2 t_1} - \tfrac{t_4 u_1 u_4 w_4^2}{2 t_1}\Big]
\end{aligned}
$$

$$t2 == \left(ul_1\, R^{-}_{2,5}\, nl_3\, ul_4\, nl_6 \ // \ m_{1,2\to1} \ // \ m_{1,3\to1} \ // \ m_{4,5\to4} \ // \ m_{4,6\to4}\right)$$

True

$$
\begin{aligned}
&z2 = R^{+}_{1,14}\, R^{-}_{5,2}\, nr_3\, ul_4\, R^{+}_{19,6}\, R^{-}_{7,10}\, nl_8\, ur_9\, R^{+}_{11,20}\, nr_{12}\, ul_{13}\, R^{-}_{15,18}\\
&\quad nl_{16}\, ur_{17};\\
&(\text{Do}[z2 = z2 \ // \ m_{1,k\to1},\ \{k, 2, 20\}];\ z2 = z2\ /.\ a\_{}_1 \mapsto a)
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\Big[-1 + \tfrac{1}{t} + t,\ 0,\ 0,\\
&\quad -16 + \tfrac{9c}{2} - \tfrac{2c}{t^4} + \tfrac{1}{t^3} + \tfrac{11c}{2t^3} - \tfrac{4}{t^2} - \tfrac{8c}{t^2} + \tfrac{10}{t} + \tfrac{4c}{t} + 18t - 10ct - 14t^2 +\\
&\quad 8ct^2 + 7t^3 - \tfrac{3ct^3}{2} - 2t^4 - 2ct^4 + 2ct^5 - \tfrac{ct^6}{2} - 4uw + \tfrac{2uw}{t^4} -\\
&\quad \tfrac{7uw}{2t^3} + \tfrac{9uw}{2t^2} + \tfrac{uw}{2t} + 6tuw - 2t^2uw - \tfrac{1}{2}t^3uw + \tfrac{3}{2}t^4uw - \tfrac{1}{2}t^5uw\Big]
\end{aligned}
$$

```
FromCoefficientRules[
 CoefficientRules[z2[[4]], {c, u, w}] /.
  {(e_ → a_) ↦ (e → Simplify[a])}, {c, u, w}]
```

$$
-\frac{\left(1-t+t^2\right)^2 \left(-1+2t-3t^2+2t^3\right)}{t^3} - \frac{c\left(1-t+t^2\right)^3 \left(4+t-5t^2-t^3+t^4\right)}{2t^4} - \frac{\left(1-t+t^2\right)^3 \left(-4-5t+t^3\right) uw}{2t^4}
$$

"God created the knots; all else in topology is the work of mortals."

Leopold Kronecker (modified) katlas.org

Questions and To Do List. • Clean up and write up. • Implement well, compute for everything in sight. • Why are our quantities polynomials rather than just rational functions? • Bounds on their degrees? • Find the 2-variable version (for knots). How complex is it? • What about links / closed components? • Fully digest the "expansion" theorem; include cuaps. • Explore the (non-)dependence on R. • Is there a canonical R? • What does "group like" mean? • Strand removal? Strand doubling? Strand reversal? • Say something about knot genus. • Find the EK/AT/KV "vertex". • Use as a playground to study associators/braidors. • Restate in topological language. • Study the associated (v-)braid representations. • Study mirror images and the $b^+ \leftrightarrow b^-$ involution. • Study ribbon knots. • Make precise the relationship with Γ-calculus and Alexander. • Relate to the coloured Jones polynomial. • Relate with "ordinary" q-algebra. • k-smidgen sl_n, etc. • Are there "solvable" CYBE algebras not arising from semi-simple algebras? • Categorify and appease the Gods.

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Disclaimer. This is all quite new. The overall picture is correct, yet some details might be somewhat off. Many pieces are certainly not in their final form yet. Help Needed!