



A Poly-Time Knot Polynomial Via Solvable Approximation

Work in Progress!

Abstract. I will construct the first poly-time-computable knot polynomial since Alexander's [AI, 1928] by using some new commutator-calculus techniques and a Lie algebra \mathfrak{g}_1 which is at the same time solvable and an approximation of the simple Lie algebra sl_2 .

Expected! Finite-type invariants include all coefficients of all quantum knot polynomials (appropriately parametrized), and each is computable in poly-time. Yet A green paradise...

d	2	3	4	5	6	7	8	...
known f.t. invts in $O(n^d)$	1	1	∞	3	4	8	11	...

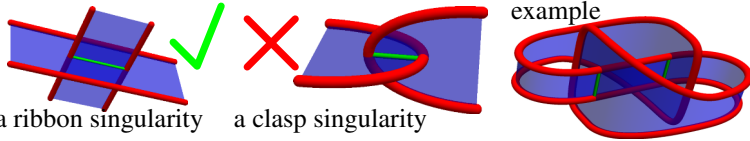
This is an unreasonable picture! So there ought to be further poly-time polynomial invariants.

Also. • The line above the Alexander line in the Melvin-Morton [MM, Ro] expansion of the coloured Jones polynomial. • The 2-loop contribution to the Kontsevich integral.



Paradise! Foremost reason: **OBVIOUSLY**. Cf. proving (incomputable A)=(incomputable B), or categorifying (incomputable C).

Secondary reason: may disprove {ribbon} = {slice}:

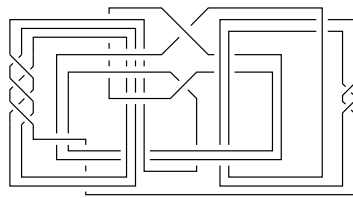


A bit about ribbon knots. A "ribbon knot" is a knot that can be presented as the boundary of a disk that has "ribbon singularities", but no "clasp singularities". A "slice knot" is a knot in $S^3 = \partial B^4$ which is the boundary of a non-singular disk in B^4 . Every ribbon knots is clearly slice, yet,

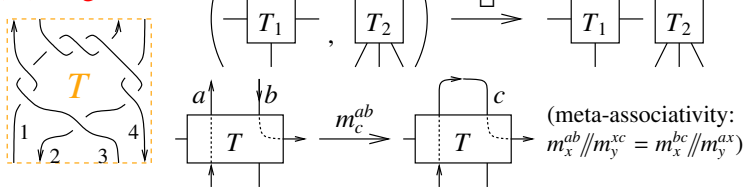
Conjecture. Some slice knots are not ribbon.

In [BN2] I list 5 criteria an invariant needs to meet to have a fair chance of detecting non-ribbons. **Ours meets all 5.**

[GST]: Gompf, Scharlemann, Thompson:
 $a_+ = -t^8 + 2t^7 - t^6 - 2t^4 + 5t^3 - 2t^2 - 7t + 13$
 $e_+ = 5t^{15} - 18t^{14} + 33t^{13} - 32t^{12} + 2t^{11} + 42t^{10} - 62t^9 - 8t^8 + 166t^7 - 242t^6 + 108t^5 + 132t^4 - 226t^3 + 148t^2 - 11t - 36$



(v-)Tangles.

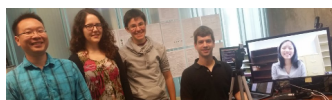


Theorem [EK, Ha, En, Se]. There is a "homomorphic expansion"

$Z: \left\{ \begin{matrix} S\text{-component} \\ (v/b)\text{-tangles} \end{matrix} \right\} \rightarrow \mathcal{A}_S^v :=$

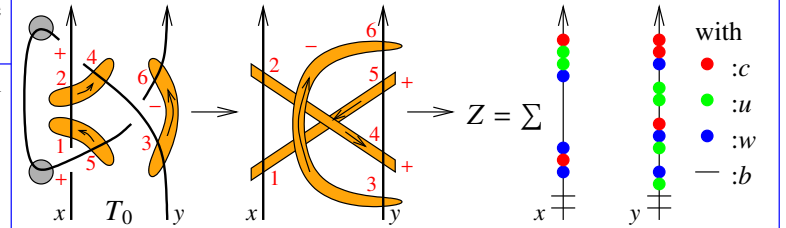
AS: $\text{Y} + \text{X} = 0$
 STU: $\text{Y} = \text{X} - \text{Z}$
 IHX: $\text{Y} = \text{X} - \text{Z}$

Videos of a 4-hour version of this talk are at œβ/LD. **Videos** of private seminar meetings are at œβ/PP.



Many thanks: Vo, Halacheva, Dalvit, Ens, Lee (van der Veen, Schaveling)

Algebras and Invariants. Given any unital algebra A (typically, $A \sim \hat{\mathcal{U}}(\mathfrak{g})$), appropriate orange $R \in A \otimes A$, and appropriate cuaps $\in A$, get an $A^{\otimes S}$ -valued invariant of pure S -component tangles:



Good News. In theory, enough to know R , the cuaps, and stitching/multiplication $m_k^{ij}: A_i \otimes A_j \rightarrow A_k$.

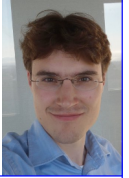
Problem. Extract information out of Z .

Textbook Solution. Use representation theory ... works, slowly.

Today's Solution. (with van der Veen) For some specific \mathfrak{g} 's, work in a space of "formulas of a specific type" for elements of $\hat{\mathcal{U}}(\mathfrak{g})^{\otimes S}$:

$$\left\{ \begin{matrix} \text{ordered perturbed} \\ \text{Gaussian formulas} \end{matrix} \right\} \rightarrow \hat{\mathcal{U}}(\mathfrak{g})^{\otimes S}$$

van der Veen



1-Smidgen sl_2 Let \mathfrak{g}_1 be the 4-dimensional Lie algebra $\mathfrak{g}_1 = \langle b, c, u, w \rangle$ over the ring $R = \mathbb{Q}[\epsilon]/(\epsilon^2 = 0)$, with b central and with $[w, c] = w$, $[c, u] = u$, and $[u, w] = b - 2\epsilon c$, with CYBE $r_{ij} = (b_i - \epsilon c_i)c_j + u_i w_j$ in $\mathcal{U}(\mathfrak{g}_1)^{\otimes [i, j]}$. Over \mathbb{Q} , \mathfrak{g}_1 is a **solvable approximation of sl_2** : $\mathfrak{g}_1 \supset \langle b, u, w, \epsilon b, \epsilon c, \epsilon u, \epsilon w \rangle \supset \langle b, \epsilon b, \epsilon c, \epsilon u, \epsilon w \rangle \supset 0$. (note: $\deg(b, c, u, w, \epsilon) = (1, 0, 1, 0, 1)$)

0-Smidgen $sl_2 \odot$. Let \mathfrak{g}_0 be \mathfrak{g}_1 at $\epsilon = 0$, or $\mathbb{Q}\langle b, c, u, w \rangle / ([b, \cdot] = 0, [c, u] = u, [c, w] = -w, [u, w] = b)$ with $r_{ij} = b_i c_j + u_i w_j$. It is $\mathfrak{b}^* \rtimes \mathfrak{b}$ where \mathfrak{b} is the 2D Lie algebra $\mathbb{Q}\langle c, w \rangle$ and (b, u) is the dual basis of (c, w) . It is even more valuable than \mathfrak{g}_1 , but topology already got by other means almost everything \mathfrak{g}_0 has to give.

How did these arise? $sl_2 = \mathfrak{b}^+ \oplus \mathfrak{b}^- / \mathfrak{h} =: sl_2^+ / \mathfrak{h}$, where $\mathfrak{b}^+ = \langle c, w \rangle / [w, c] = w$ is a Lie bialgebra with $\delta: \mathfrak{b}^+ \rightarrow \mathfrak{b}^+ \otimes \mathfrak{b}^+$ by $\delta: (c, w) \mapsto (0, c \wedge w)$. Going back, $sl_2^+ = \mathcal{D}(\mathfrak{b}^+) = (\mathfrak{b}^+)^* \oplus \mathfrak{b}^+ = \langle b, u, c, w \rangle / \dots$. **Idea.** Replace $\delta \rightarrow \epsilon \delta$ over $\mathbb{Q}[\epsilon]/(\epsilon^{k+1} = 0)$. At $k = 0$, get \mathfrak{g}_0 . At $k = 1$, get $[w, c] = w$, $[w, b'] = -\epsilon w$, $[c, u] = u$, $[b', u] = -\epsilon u$, $[b', c] = 0$, and $[u, w] = b' - \epsilon c$. Now note that $b' + \epsilon c$ is central, so switch to $b := b' + \epsilon c$. This is \mathfrak{g}_1 .

Ordering Symbols. \odot (*poly* | *specs*) plants the variables of *poly* in $S(\oplus_i \mathfrak{g}_i)$ on several tensor copies of $\mathcal{U}(\mathfrak{g})$ according to *specs*. E.g., $\odot(c^3 u_1 c_2 e^{u_3} w_3^9 | x: w_3 c_1, y: u_1 u_3 c_2) = w^9 c^3 \odot u e^u c \in \mathcal{U}(\mathfrak{g})_x \otimes \mathcal{U}(\mathfrak{g})_y$

This enables the description of elements of $\hat{\mathcal{U}}(\mathfrak{g})^{\otimes S}$ using commutative polynomials / power series.

0-Smidgen Invariants. $r = Id \in \mathfrak{b}^- \otimes \mathfrak{b}^+$ solves the CYBE $[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$ in $\mathcal{U}(\mathfrak{g}_0)^{\otimes 3}$ and, by luck,

$$\frac{+}{i} \frac{-}{j} = \frac{+}{i} \frac{+}{j} = R_{ij} = e^{r_{ij}} = e^{b_i c_j + u_i w_j} \in \mathcal{U}(\mathfrak{g}_{0,i} \oplus \mathfrak{g}_{0,j})$$

solves YB/R3.

Lemma. $R_{ij} = e^{b_i c_j + u_i w_j} = \odot \left(\exp(b_i c_j + \frac{e^{b_i - 1}}{b_i} u_i w_j) \mid i: u_i, j: c_j w_j \right)$

Example. $Z(T_0) = \sum_{m, n} \frac{b_i^{m-n} (e^{b_i - 1})^n}{m! n!} u^m \otimes c^m w^n$

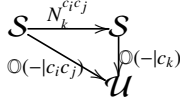
$$\odot \left(\exp(b_5 c_1 + \frac{e^{b_5 - 1}}{b_5} u_5 w_1 + b_2 c_4 + \frac{e^{b_2 - 1}}{b_2} u_2 w_4 - b_3 c_6 + \frac{e^{-b_3 - 1}}{b_3} u_3 w_6) \mid x: c_1 w_1 u_2, y: u_3 c_4 w_4 u_5 c_6 w_6 \right) = \odot(? | x: c_x u_x w_x, y: c_y u_y w_y)$$

"cuw form"

Goal. Write ? as a Gaussian: ωe^{L+Q} where L bilinear in b_i and c_i with integer coefficients, Q a balanced quadratic in u_i and w_i with coefficients in $R_S := \mathbb{Q}(b_i, e^{b_i})$, and $\omega \in R_S$.

The Big g_0 Lemma. Under $[c, u] = u$, $[c, w] = -w$, and $[u, w] = b$:

1. $N_k^{c_i c_j} := \mathbb{O}(\zeta | c_i c_j) \stackrel{\cong}{=} \mathbb{O}(\zeta / (c_i, c_j \rightarrow c_k) | c_k)$
(Meaning, $N_k^{c_i c_j} : \zeta \mapsto (\zeta / (c_i, c_j \rightarrow c_k))$) and the diagram commutes. Trivial, also for b, u, w .
- 2a. $N^{uc} := \mathbb{O}(e^{\gamma c + \beta u} | uc) \stackrel{\cong}{=} \mathbb{O}(e^{\gamma c + e^{-\gamma} \beta u} | cu)$ (means $e^{\beta u} e^{\gamma c} = e^{\gamma c} e^{-\gamma \beta u}$)
- 2b. $N^{wc} := \mathbb{O}(e^{\gamma c + \alpha w} | wc) \stackrel{\cong}{=} \mathbb{O}(e^{\gamma c + e^{\gamma} \alpha w} | cw)$... in the $\{ax + b\}$ group)
3. $\mathbb{O}(e^{\alpha w + \beta u} | wu) = \mathbb{O}(e^{-\beta \alpha \beta + \alpha w + \beta u} | uw)$ (the Weyl relations)
4. $\mathbb{O}(e^{\delta u w} | wu) e^{\beta u} = e^{\gamma \beta u} \mathbb{O}(e^{\delta u w} | wu)$, with $\gamma = (1 + b\delta)^{-1}$
- (a. expand and crunch. b. use $w = b\hat{x}$, $u = \partial_x$. c. use "scatter and glow".)
5. $\mathbb{O}(e^{\delta u w} | wu) = \mathbb{O}(v e^{\gamma \delta u w} | uw)$ (same techniques)
6. $N^{wu} := \mathbb{O}(e^{\beta u + \alpha w + \delta u w} | wu) \stackrel{\cong}{=} \mathbb{O}(v e^{-\beta \alpha \beta + \gamma \alpha w + \gamma \beta u + \gamma \delta u w} | uw)$



Sneaky. α may contain (other) u 's, β may contain (other) w 's.

Strand Stitching. m_k^{ij} , is defined as the composition

$$c_i u_i \overline{w_i} c_j u_j w_j \xrightarrow{N_k^{w_i c_j}} c_i \overline{u_i} c_k \overline{w_k} u_j w_j \xrightarrow{N_k^{u_i c_k} // N_k^{w_k u_j}} c_i c_k \overline{u_k} u_k \overline{w_k} w_j \xrightarrow{N_k^{c_i c_k} // - // N_k^{w_k w_j}} c_k u_k w_k$$

Γ -calculus [BNS, BN1]. After re-packaging, especially setting $t_i := e^{b_i}$, an S -component tangle T has $\Gamma(T) \in R_S \times M_{S \times S}(R_S) = \left\{ \begin{array}{c|c} \omega & S \\ \hline S & A \end{array} \right\}$ with $R_S := \mathbb{Z}(\{t_a : a \in S\})$:

$$\left(\begin{array}{c|c} a & b \\ \hline b & a \end{array} \right) \rightarrow \begin{array}{c|cc} 1 & a & b \\ \hline a & 1 - t_a^{\pm 1} & \\ b & 0 & t_a^{\pm 1} \end{array} \quad T_1 \sqcup T_2 \rightarrow \begin{array}{c|cc} \omega_1 \omega_2 & S_1 & S_2 \\ \hline S_1 & A_1 & 0 \\ S_2 & 0 & A_2 \end{array}$$

$$\begin{array}{c|ccc} \omega & a & b & S \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi \end{array} \xrightarrow{m_c^{ab}} \left(\begin{array}{c|cc} (1-\beta)\omega & c & S \\ \hline c & \gamma + \frac{\alpha\delta}{1-\beta} & \epsilon + \frac{\delta\theta}{1-\beta} \\ S & \phi + \frac{\alpha\psi}{1-\beta} & \Xi + \frac{\psi\theta}{1-\beta} \end{array} \right)$$

For long knots, ω is Alexander, and that's the fastest Alexander algorithm I know!
Dunfield: 1000-crossing fast.



1-Smidgen Invariants. Much is the same:

The Big g_1 Lemma. Parts 1 and 2 are the same, yet

6. $\mathbb{O}(e^{\alpha w + \beta u + \delta u w} | wu) = \mathbb{O}(v(1 + \epsilon v \Lambda) e^{\nu(-\beta \alpha \beta + \alpha w + \beta u + \delta u w)} | cuw)$
Here Λ is for $\Lambda\delta\gamma\sigma\varsigma$, "a principle of order and knowledge", a balanced quartic in α, β, c, u , and w :

$$\Lambda = -bv \left(v^2 \alpha^2 \beta^2 + 4\delta v \alpha \beta + 2\delta^2 \right) / 2 - \delta v^3 (3b\delta + 2)\beta^2 u^2 / 2 - b\delta^4 v^3 u^2 w^2 / 2 - \delta^2 v^3 (2b\delta + 1)\beta u^2 w - v^2 (2b\delta + 1)(v\alpha\beta + 2\delta)\beta u - 2b\delta^2 v^2 (v\alpha\beta + \delta)uw + \delta v^3 (b\delta + 2)\alpha^2 w^2 / 2 + 2(v\alpha\beta + \delta)c + 2\delta v \beta c u + 2\delta^2 \gamma c u w + 2\delta v \alpha c w + \delta^2 v^3 \alpha u w^2 + v^2 (v\alpha\beta + 2\delta)\alpha w.$$

Proof. A brutal hell.

Problem. We now need to normal-order perturbed Gaussians!

Solution. Borrow some tactics from QFT:

$$\mathbb{O}(\epsilon P(c, u) e^{\gamma c + \beta u} | uc) = \mathbb{O}(\epsilon P(\partial_\gamma, \partial_\beta) e^{\gamma c + \beta u} | uc) = \mathbb{O}(\epsilon P(\partial_\gamma, \partial_\beta) e^{\gamma c + e^{-\gamma} \beta u} | cu),$$

and likewise

$$\mathbb{O}(\epsilon P(u, w) e^{\alpha w + \beta u + \delta u w} | wu) = \mathbb{O}(\epsilon P(\partial_\beta, \partial_\alpha) v e^{\nu(-\beta \alpha \beta + \alpha w + \beta u + \delta u w)} | cuw)$$

Note. Strand stitching requires a tiny extra step.

Finally, the values of the generators $\overleftarrow{\nearrow}, \overrightarrow{\searrow}, \overrightarrow{n}, \overleftarrow{n}, \overrightarrow{u},$ and \overleftarrow{u} , are set by brutally solving many equations, non-uniquely.

Pragmatic Simplifications. Get rid of $\zeta = (e^b - 1)/b$ factors by rescaling $u \rightarrow \bar{u} = \zeta u$. Complement this with $\beta \rightarrow \bar{\beta} = \zeta^{-1} \beta$, $\delta \rightarrow \bar{\delta} = \zeta^{-1} \delta$, $\epsilon \rightarrow \bar{\epsilon} = \zeta^{-1} \epsilon$. Simplify further by naming $e^b \rightarrow t$; e.g., $v \rightarrow \bar{v} = (1 + (t-1)\delta)^{-1}$. Get confused by renaming $(\bar{u}, \bar{\beta}, \bar{\delta}, \bar{v}) \rightarrow (u, \beta, \delta, v)$, and more confused by working with $\mu = v^{-1}$ and $\mathbb{E}(\omega, L, Q, P) := \omega^{-1} (1 + \epsilon \omega^{-4} P) e^{L + \omega^{-1} Q}$, where $\omega \in R := \mathbb{Q}(t_k)$, $L = \sum l_{ij} b_i c_j$ with $l_{ij} \in \mathbb{Z}$, $Q = \sum q_{ij} u_i w_j$ with $q_{ij} \in R$, and P is a balanced quartic polynomial in c_i, u_i , and w_i with coefficients in R . Magically, all coefficients are now Laurent polynomials in the t_k 's.

Rough complexity estimate, after $t_k \rightarrow t$: n : xing number; w : width, maybe $\frac{n}{A} \sum_{d=0}^4 \frac{w^{4-d}}{E} \frac{w^d}{F} \frac{n^2}{G} = n^3 w^4 \in [n^5, n^7]$
 $\sim \sqrt{n}$. A : go over stitchings in order. B : multiplication ops per $N^{u_i w_j}$. d : deg of u_i, w_j in P . E : #terms of deg d in P . F : ops per term. G : cost per polynomial multiplication op.

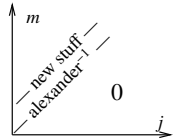
Expectation. Our invariant is the "1-higher diagonal" in the MMR expansion of the coloured Jones polynomial J_λ .



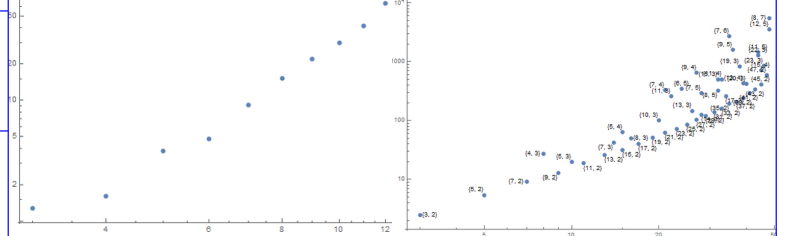
Theorem ([BNG], conjectured [MM], elucidated [Ro]). Let $J_d(K)$ be the coloured Jones polynomial of K , in the d -dimensional representation of $sl(2)$. Writing

$$\left. \frac{(q^{1/2} - q^{-1/2}) J_d(K)}{q^{d/2} - q^{-d/2}} \right|_{q=e^h} = \sum_{j,m \geq 0} a_{jm}(K) d^j h^m,$$

"below diagonal" coefficients vanish, $a_{jm}(K) = 0$ if $j > m$, and "on diagonal" coefficients give the inverse of the Alexander polynomial: $(\sum_{m=0}^{\infty} a_{mm}(K) h^m) \cdot A(K) (e^h) = 1$.



Experimental Analysis ($\omega\epsilon\beta$ /Exp). Log-log plot of computation time (sec) vs. crossing number, for all knots with up to 12 crossings (mean times) and for all torus knots with up to 48 crossings:



Conjecture (checked on the same collections). Given a knot K with Alexander polynomial a , there is a polynomial e (the "essence" of P) such that

$$P = a^2 \left((t - 2 + t^{-1})e + t a d' \left(\frac{(4 + t - t^2)(uw + (t-1)c)}{2(t-1)} - 1 \right) \right).$$

Furthermore, a and e are symmetric under $t \rightarrow t^{-1}$, so let a_+ and e_+ be their "positive parts", so e.g., $e(t) = e_+(t) + e_+(t^{-1}) - e_+(0)$.

Power. On the 250 knots with at most 10 crossings, the pair (a, e) attains 250 distinct values, while (Khovanov, HOMFLYPT) attains only 249 distinct values. To 11 crossings the numbers are (802, 788, 772) and to 12 they are (2978, 2883, 2786).

Genus. Up to 12 crossings, always $\deg e_+ \leq 2g - 1$, where g is the 3-genus of K (equality for 2530 knots). This gives a lower bound on g in terms of e (conjectural, but undoubtedly true). This bound is often weaker than the Alexander bound, yet for 10 of the 12-crossing Alexander failures it does give the right answer.

$$R_{\theta, i, j}^+ := \mathbb{E} [b_i c_j + b_i^{-1} (e^{b_i} - 1) u_i w_j];$$

$$R_{\theta, i, j}^- := \mathbb{E} [-b_i c_j + b_i^{-1} (e^{-b_i} - 1) u_i w_j];$$

The R-matrices

CF[ω. E[Q_]] := Simplify[ω] E[Simplify[Q]]; Utilities

E /: E[Q1_] E[Q2_] := CF@E[Q1 + Q2];
 ω1. E[Q1_] ≡ ω2. E[Q2_] := Simplify[ω1 == ω2 ∧ Q1 == Q2];

Nu_i-c_j->k [ω. E[Q_]] := CF Normal Ordering Operators

ω E[e^{-γ} β u_k + γ c_k + (Q / . c_j | u_i → θ)] / . {γ → ∂_{c_j} Q, β → ∂_{u_i} Q};
 Nu_i-c_j->k [ω. E[Q_]] := CF
 ω E[e^γ α w_k + γ c_k + (Q / . c_j | w_i → θ)] / . {γ → ∂_{c_j} Q, α → ∂_{w_i} Q};
 Nu_i-u_j->k [ω. E[Q_]] := CF
 v ω E[-b_R v α β + v β u_k + v δ u_R w_k + v α w_k + (Q / . w_i | u_j → θ)] / .
 v → (1 + b_k δ)⁻¹ / .
 {α → ∂_{w_i} Q / . u_j → θ, β → ∂_{u_j} Q / . w_i → θ, δ → ∂_{w_i, u_j} Q};

m_i->j [ω. E[Q_]] := CF Stitching
 Module[{x},
 (ω E[Q] / . b_i | j → b_R // Nu_i c_j → x // Nu_i c_x → x // Nu_{w_x} u_j → x) / .
 {c_i → c_k, w_j → w_k, y_{-x} → y_k}]

Some calculations for T₀

$$T_{\theta, \theta} = R_{\theta, 5, 1}^+ R_{\theta, 2, 4}^+ R_{\theta, 3, 6}^-$$

$$\mathbb{E} \left[b_5 c_1 + b_2 c_4 - b_3 c_6 + \frac{(-1+e^{b_5}) u_5 w_1}{b_5} + \frac{(-1+e^{b_2}) u_2 w_4}{b_2} + \frac{(-1+e^{-b_3}) u_3 w_6}{b_3} \right]$$

$$T_{\theta, 1} = T_{\theta, \theta} // Nu_3 c_4 \rightarrow 4$$

$$\mathbb{E} \left[b_5 c_1 + b_2 c_4 - b_3 c_6 + \frac{(-1+e^{b_5}) u_5 w_1}{b_5} + \frac{(-1+e^{b_2}) u_2 w_4}{b_2} + \frac{e^{-b_2} (-1+e^{-b_3}) u_4 w_6}{b_3} \right]$$

$$T_{\theta, 2} = T_{\theta, 1} // Nu_4 u_5 \rightarrow 4$$

$$\mathbb{E} \left[b_5 c_1 + b_2 c_4 + \frac{(-1+e^{b_5}) (-(-1+e^{b_2}) b_4 u_2 + b_2 u_4) w_1}{b_2 b_5} + \frac{(-1+e^{b_2}) u_2 w_4}{b_2} - \frac{b_3^2 c_6 + e^{-b_2-b_3} (-1+e^{b_3}) u_4 w_6}{b_3} \right]$$

$$T_{\theta, 2} // Nu_{w_1} u_2 \rightarrow 1$$

$$\frac{1}{1 - \frac{(-1+e^{b_2}) (-1+e^{b_5}) b_1 b_4}{b_2 b_5}} \mathbb{E} \left[\frac{1}{b_3 ((-1+e^{b_2}) (-1+e^{b_5}) b_1 b_4 - b_2 b_5)} \right]$$

$$(b_3 b_5 ((-1 + e^{b_2}) (-1 + e^{b_5}) b_1 b_4 - b_2 b_5) c_1 + b_2 b_3 ((-1 + e^{b_2}) (-1 + e^{b_5}) b_1 b_4 - b_2 b_5) c_4 + (-1 + e^{b_2}) (-1 + e^{b_5}) b_3 b_4 u_1 w_1 - (-1 + e^{b_5}) b_2 b_3 u_4 w_1 - (-1 + e^{b_2}) b_3 b_5 u_1 w_4 + (-1 + e^{b_2}) (-1 + e^{b_5}) b_1 b_3 u_4 w_4 - ((-1 + e^{b_2}) (-1 + e^{b_5}) b_1 b_4 - b_2 b_5) (b_3^2 c_6 + e^{-b_2-b_3} (-1 + e^{b_3}) u_4 w_6))]$$

$$T_{\theta, \theta} // m_{1,2 \rightarrow 1} // m_{3,4 \rightarrow 3} // m_{3,5 \rightarrow 3} // m_{3,6 \rightarrow 3}$$

$$\frac{1}{1 - \frac{(-1+e^{b_1}) (-1+e^{b_3})}{(-1+e^{b_3})}} \mathbb{E} \left[b_3 c_1 + b_1 c_3 - b_3 c_3 + \frac{(-1+e^{b_1}) (-1+e^{b_3}) u_1 w_1}{(-e^{b_1} - e^{b_3} + e^{b_1+b_3}) b_1} - \frac{e^{-b_3} (-1+e^{b_1}) (b_3 u_1 - e^{b_3} (-1+e^{b_3}) b_1 u_3) w_3}{(-e^{b_1} - e^{b_3} + e^{b_1+b_3}) b_1 b_3} + \frac{e^{-b_1} (-1+e^{b_3}) u_3 (-e^{b_1+b_3} w_1 + (e^{b_1+e^{b_3}} - e^{b_1+b_3}) w_3)}{(-e^{b_1} - e^{b_3} + e^{b_1+b_3}) b_3} \right]$$

Verifying meta-associativity

$$Q\theta = \mathbb{E} [\text{Sum}[f_i c_i, \{i, 3\}] + \text{Sum}[f_{i,j} u_i w_j, \{i, 3\}, \{j, 3\}]]$$

$$\mathbb{E} [c_1 f_1 + c_2 f_2 + c_3 f_3 + u_1 w_1 f_{1,1} + u_1 w_2 f_{1,2} + u_1 w_3 f_{1,3} + u_2 w_1 f_{2,1} + u_2 w_2 f_{2,2} + u_2 w_3 f_{2,3} + u_3 w_1 f_{3,1} + u_3 w_2 f_{3,2} + u_3 w_3 f_{3,3}]$$

$$(Q\theta // m_{1,2 \rightarrow 1} // m_{1,3 \rightarrow 1}) \equiv (Q\theta // m_{2,3 \rightarrow 2} // m_{1,2 \rightarrow 1})$$

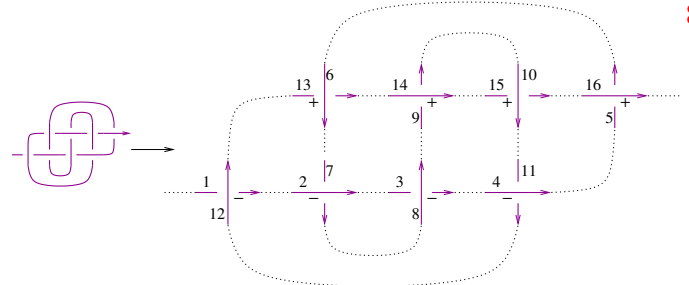
True

$$t1 = R_{\theta, 1, 2}^+ R_{\theta, 3, 4}^+ R_{\theta, 5, 6}^+ // m_{3,5 \rightarrow x} // m_{1,6 \rightarrow y} // m_{2,4 \rightarrow z}$$

$$\mathbb{E} \left[b_x (c_y + c_z) + \frac{(-1+e^{b_x}) u_x (w_y + w_z)}{b_x} + \frac{b_y^2 c_z + (-1+e^{b_y}) u_y w_z}{b_y} \right]$$

$$t1 \equiv (R_{\theta, 1, 2}^+ R_{\theta, 3, 4}^+ R_{\theta, 5, 6}^+ // m_{1,3 \rightarrow x} // m_{2,5 \rightarrow y} // m_{4,6 \rightarrow z})$$

True



817

$$z1 = R_{\theta, 12, 1}^- R_{\theta, 2, 7}^- R_{\theta, 8, 3}^- R_{\theta, 4, 11}^- R_{\theta, 16, 5}^- R_{\theta, 6, 13}^- R_{\theta, 14, 9}^- R_{\theta, 10, 15}^-$$

$$\text{Do}[z1 = (z1 // m_{1, n \rightarrow 1}) / . b_- \rightarrow b, \{n, 2, 16\}];$$

$$\{\text{CF}@z1, \text{KnotData}[\{8, 17\}, \text{"AlexanderPolynomial"}][t]\}$$

$$\left\{ -\frac{e^{3b} \mathbb{E}[\theta]}{1-4e^{b+8}e^{2b-11}e^{3b+8}e^{4b-4}e^{5b+6}e^{6b}}, 11 - \frac{1}{t^3} + \frac{4}{t^2} - \frac{8}{t} - 8t + 4t^2 - t^3 \right\}$$

Demo Programs for 1-Co. ωβ/Demo

$$\Delta[k_-] := (1 - t_k) (\alpha^2 \beta^2 + 4 \alpha \beta \delta \mu + 2 \delta^2 \mu^2) / 2 + 2 \mu^2 (\alpha \beta + \delta \mu) c_k - \beta (2 \mu - 1) (\alpha \beta + 2 \delta \mu) u_k + 2 \beta \delta \mu^2 c_k u_k - \beta^2 \delta (3 \mu - 1) u_k^2 / 2 + \alpha (\alpha \beta + 2 \delta \mu) w_k + 2 \alpha \delta \mu^2 c_k w_k - 2 (t_k - 1) \delta^2 (\alpha \beta + \delta \mu) u_k w_k + 2 \delta^2 \mu^2 c_k u_k w_k - \beta \delta^2 (2 \mu - 1) u_k^2 w_k + \alpha^2 \delta (1 + \mu) w_k^2 / 2 + \alpha \delta^2 u_k w_k^2 - (t_k - 1) \delta^4 u_k^2 w_k^2 / 2;$$

The Λόγος

Differential Polynomials

DP_{x → D_α, y → D_β} [P_] [f_] := (* means P[∂_α, ∂_β][f] *)
 Total[CoefficientRules[P, {x, y}] / .
 ({m_, n_} → c_) ⇒ c D[f, {α, m}, {β, n}]]

CF[E[ω, L_, Q_, P_]] := Expand/@Together/@
 E[ω / . b_L ⇒ Log[t_L], L, Q / . b_L ⇒ Log[t_L],
 P / . b_L ⇒ Log[t_L]];

E /: E[ω1_, L1_, Q1_, P1_] E[ω2_, L2_, Q2_, P2_] :=
 CF@E[ω1 ω2, L1 + L2, ω2 Q1 + ω1 Q2, ω2⁴ P1 + ω1⁴ P2];

Utilities

Normal Ordering Operators

Nu_i-c_j->k [E[ω, L_, Q_, P_]] := With[{q = e^{-γ} β u_k + γ c_k}, CF[
 E[ω, γ c_k + (L / . c_j → θ), ω e^{-γ} β u_k + (Q / . u_i → θ),
 e^{-q} DP_{c_j → D_γ, u_i → D_β} [P] [e^q]] / . {γ → ∂_{c_j} L, β → ω⁻¹ ∂_{u_i} Q}];

Nu_i-c_j->k [E[ω, L_, Q_, P_]] := With[{q = e^γ α w_k + γ c_k}, CF[
 E[ω, γ c_k + (L / . c_j → θ), ω e^γ α w_k + (Q / . w_i → θ),
 e^{-q} DP_{c_j → D_γ, w_i → D_α} [P] [e^q]] / . {γ → ∂_{c_j} L, α → ω⁻¹ ∂_{w_i} Q}];

Nu_i-u_j->k [E[ω, L_, Q_, P_]] :=
 With[{q = (1 - t_k) μ⁻¹ α β + μ⁻¹ β u_k + μ⁻¹ δ u_k w_k + μ⁻¹ α w_k}, CF[
 E[μ ω, L, μ ω q + μ (Q / . w_i | u_j → θ),
 μ⁴ e^{-q} DP_{w_i → D_α, u_j → D_β} [P] [e^q] + ω⁴ Δ[k]] / .
 μ → 1 + (t_k - 1) δ / .
 {α → ω⁻¹ (∂_{w_i} Q / . u_j → θ), β → ω⁻¹ (∂_{u_j} Q / . w_i → θ),
 δ → ω⁻¹ ∂_{w_i, u_j} Q}];

Stitching

m_i->j [Z_] := Module[{x, y, z},
 Z // Nu_{w_i} c_j → x // Nu_{w_x} u_j → y // ReplaceAll[{c_{x|y} → c_x, w_j → w_y}] //
 Nu_{c_x} // ReplaceAll[Z_{-i|j|x|y} → Z_k] // CF]

The Generators

$$R_{i,j}^+ := \mathbb{E} \left[1, b_i c_j, u_i w_j, -c_i (t_i - 1)^2 / 2 - c_i^2 (t_i - 1)^2 / 2 + c_i c_j (t_j^2 - t_i - 2) / 2 - c_j u_i w_i / 2 + c_i (1 - t_i) u_i w_i - u_i^2 w_i^2 / 2 + u_i w_j + c_j t_i u_i w_j / 2 + c_i (t_i - 2) t_i u_i w_j + c_i (1 + t_j) u_j w_j / 2 + (t_i - 1) u_i^2 w_i w_j - (t_i - 2) t_i u_i^2 w_j^2 / 2 \right];$$

$$R_{i,j}^- := \mathbb{E} \left[1, -b_i c_j, -t_i^{-1} u_i w_j, c_i (t_i - 1)^2 / 2 + c_i^2 (t_i - 1)^2 / 2 + c_i c_j (2 + t_i - t_j^2) / 2 + c_j u_i w_i / 2 + c_i (t_i - 1) u_i w_i + u_i^2 w_i^2 / 2 + (1 - t_i^{-1}) u_i w_j / 2 + c_i (2 t_i - 5 + 3 t_i^{-1}) u_i w_j / 2 + c_j (t_i^{-1} + 1 - t_i^{-1} t_j^2) u_i w_j / 2 - c_i (t_j + 1) u_j w_j / 2 + (2 - 3 t_i^{-1}) u_i^2 w_i w_j / 2 + (1 + 2 t_i^{-2} - 3 t_i^{-1}) u_i^2 w_j^2 / 2 - t_i^{-1} (1 + t_j) u_i u_j w_j^2 / 2 \right];$$

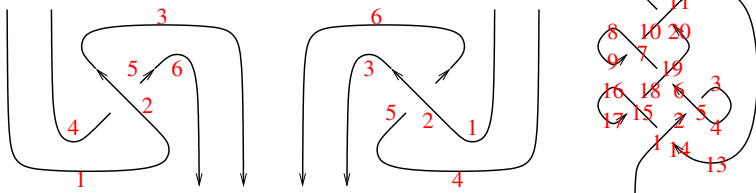
$$ur_i := \mathbb{E} \left[t_i^{-1/4}, 0, 0, c_i t_i / 4 + u_i w_i / 8 \right];$$

$$nr_i := \mathbb{E} \left[t_i^{1/4}, 0, 0, -c_i t_i^3 / 4 - t_i^2 u_i w_i / 8 \right];$$

$$ul_i := \mathbb{E} \left[t_i^{1/4}, 0, 0, c_i t_i (4 + t_i) / 4 - t_i^2 u_i w_i / 8 \right];$$

$$nl_i := \mathbb{E} \left[t_i^{-1/4}, 0, 0, -c_i (1 + 4 t_i^{-1}) / 4 + u_i w_i / 8 \right];$$

Two swirls and a 0-framed trefoil



$$t2 = ur_1 R_{2,5} nr_3 ur_4 nr_6 // m_{1,2 \rightarrow 1} // m_{1,3 \rightarrow 1} // m_{4,5 \rightarrow 4} // m_{4,6 \rightarrow 4}$$

$$\mathbb{E} \left[1, -b_1 c_4, -\frac{u_1 w_4}{t_1}, \frac{c_1}{2} + \frac{c_1^2}{2} + c_1 c_4 - c_1 t_1 - c_1^2 t_1 + \frac{1}{2} c_1 c_4 t_1 + \frac{1}{2} c_1 t_1^2 + \frac{1}{2} c_1^2 t_1^2 - \frac{1}{2} c_1 c_4 t_4^2 - c_1 u_1 w_1 + \frac{1}{2} c_4 u_1 w_1 + c_1 t_1 u_1 w_1 + \frac{1}{2} u_1^2 w_1^2 + \frac{3 u_1 w_4}{8} - \frac{5}{2} c_1 u_1 w_4 + \frac{1}{2} c_4 u_1 w_4 - \frac{u_1 w_4}{2 t_1} + \frac{3 c_1 u_1 w_4}{2 t_1} + \frac{c_4 u_1 w_4}{2 t_1} - \frac{1}{8} t_1 u_1 w_4 + c_1 t_1 u_1 w_4 + \frac{t_4 u_1 w_4}{8 t_1} + \frac{t_4^2 u_1 w_4}{8 t_1} - \frac{c_4 t_4^2 u_1 w_4}{2 t_1} - \frac{1}{2} c_1 u_4 w_4 - \frac{1}{2} c_1 t_4 u_4 w_4 + u_1^2 w_1 w_4 - \frac{3 u_1^2 w_1 w_4}{2 t_1} + \frac{1}{2} u_1^2 w_4^2 + \frac{u_1^2 w_4^2}{t_1^2} - \frac{3 u_1^2 w_4^2}{2 t_1} - \frac{u_1 u_4 w_4^2}{2 t_1} - \frac{t_4 u_1 u_4 w_4^2}{2 t_1} \right]$$

$$t2 = (ul_1 R_{2,5} nl_3 ul_4 nl_6 // m_{1,2 \rightarrow 1} // m_{1,3 \rightarrow 1} // m_{4,5 \rightarrow 4} // m_{4,6 \rightarrow 4})$$

True

$$z2 = R_{1,14}^+ R_{5,2} nr_3 ul_4 R_{19,6}^+ R_{7,10}^- nl_8 ur_9 R_{11,20}^- nr_{12} ul_{13} R_{15,18}^- nl_{16} ur_{17};$$

$$(Do[z2 = z2 // m_{1,k \rightarrow 1}, \{k, 2, 20\}]; z2 = z2 /. a_{-1} \to a)$$

$$\mathbb{E} \left[-1 + \frac{1}{t} + t, 0, 0, -16 + \frac{9c}{2} - \frac{2c}{t^4} + \frac{1}{t^3} + \frac{11c}{2t^3} - \frac{4}{t^2} - \frac{8c}{t^2} + \frac{10}{t} + \frac{4c}{t} + 18t - 10ct - 14t^2 + 8ct^2 + 7t^3 - \frac{3ct^3}{2} - 2t^4 - 2ct^4 + 2ct^5 - \frac{ct^6}{2} - 4uw + \frac{2uw}{t^4} - \frac{7uw}{2t^3} + \frac{9uw}{2t^2} + \frac{uw}{2t} + 6t uw - 2t^2 uw - \frac{1}{2} t^3 uw + \frac{3}{2} t^4 uw - \frac{1}{2} t^5 uw \right]$$

FromCoefficientRules[

$$\text{CoefficientRules}[z2[[4]], \{c, u, w\}] /.$$

$$\{(e_{-} \to a_{-}) \Rightarrow (e \to \text{Simplify}[a])\}, \{c, u, w\}$$

$$-\frac{(1-t+t^2)^2 (-1+2t-3t^2+2t^3)}{t^3} - \frac{c(1-t+t^2)^3 (4+5t^2-t^3+t^4)}{2t^4} - \frac{(1-t+t^2)^3 (-4-5t+t^3)uw}{2t^4}$$

Questions and To Do List. • Clean up and write up. • Implement well, compute for everything in sight. • Why are our quantities polynomials rather than just rational functions? • Bounds on their degrees? • Their integrality (\mathbb{Z}) properties? • Can everything be re-stated using integrals (\int)? • Find the 2-variable version (for knots). How complex is it? • What about links / closed components? • Fully digest the “expansion” theorem; include cuaps. • Explore the (non-)dependence on R . • Is there a canonical R ? • What does “group like” mean? • Strand removal? Strand doubling? Strand reversal? • Say something about knot genus. • Find the EK/AT/KV “vertex”. • Use as a playground to study associators/braidors. • Restate in topological language. • Study the associated (v-)braid representations. • Study mirror images and the $b^+ \leftrightarrow b^-$ involution. • Study ribbon knots. • Make precise the relationship with Γ -calculus and Alexander. • Relate to the coloured Jones polynomial. • Relate with “ordinary” q -algebra. • k -smidgen sl_n , etc. • Are there “solvable” CYBE algebras not arising from semi-simple algebras? • Categorify and appease the Gods.

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Disclaimer. This is all quite new. The overall picture is correct, yet some details might be somewhat off. Many pieces are certainly not in their final form yet. **Help Needed!**

diagram	n_k^i	Alexander a_+	genus / ribbon	diagram	n_k^i	Alexander a_+	genus / ribbon
			unknotting number / amphicheiral				unknotting number / amphicheiral
	0_1^a	1	0 / ✓ 0 / ✓		3_1^a	$t - 1$	1 / ✗ 1 / ✗
	4_1^a	$3 - t$	1 / ✗ 1 / ✓		5_1^a	$t^2 - t + 1$	2 / ✗ 2 / ✗

diagram	n_k^l Alexander a_+ P -essence e_+	genus / ribbon unknotting number / amphicheiral	diagram	n_k^l Alexander a_+ P -essence e_+	genus / ribbon unknotting number / amphicheiral
	5_2^a $2t - 3$ $5t - 4$	1 / ✗ 1 / ✗		6_1^a $5 - 2t$ $t - 4$	1 / ✓ 1 / ✗
	6_2^a $-t^2 + 3t - 3$ $t^3 - 4t^2 + 4t - 4$	2 / ✗ 1 / ✗		6_3^a $t^2 - 3t + 5$ 0	2 / ✗ 1 / ✓
	7_1^a $t^3 - t^2 + t - 1$ $3t^5 + 5t^3 + 6t$	3 / ✗ 3 / ✗		7_2^a $3t - 5$ $14t - 16$	1 / ✗ 1 / ✗
	7_3^a $2t^2 - 3t + 3$ $-9t^3 + 8t^2 - 16t + 12$	2 / ✗ 2 / ✗		7_4^a $4t - 7$ $32 - 24t$	1 / ✗ 2 / ✗
	7_5^a $2t^2 - 4t + 5$ $9t^3 - 16t^2 + 29t - 28$	2 / ✗ 2 / ✗		7_6^a $-t^2 + 5t - 7$ $t^3 - 8t^2 + 19t - 20$	2 / ✗ 1 / ✗
	7_7^a $t^2 - 5t + 9$ $8 - 3t$	2 / ✗ 1 / ✗		8_1^a $7 - 3t$ $5t - 16$	1 / ✗ 1 / ✗
	8_5^a $-t^3 + 3t^2 - 3t + 3$ $2t^5 - 8t^4 + 10t^3 - 12t^2 + 13t - 12$	3 / ✗ 2 / ✗		8_3^a $9 - 4t$ 0	1 / ✗ 2 / ✓
	8_4^a $-2t^2 + 5t - 5$ $3t^3 - 8t^2 + 6t - 4$	2 / ✗ 2 / ✗		8_5^a $-t^3 + 3t^2 - 4t + 5$ $-2t^5 + 8t^4 - 13t^3 + 20t^2 - 22t + 24$	3 / ✗ 2 / ✗
	8_6^a $-2t^2 + 6t - 7$ $5t^3 - 20t^2 + 28t - 32$	2 / ✗ 2 / ✗		8_7^a $t^3 - 3t^2 + 5t - 5$ $-t^5 + 4t^4 - 10t^3 + 12t^2 - 13t + 12$	3 / ✗ 1 / ✗
	8_8^a $2t^2 - 6t + 9$ $-t^3 + 4t^2 - 12t + 16$	2 / ✓ 2 / ✗		8_9^a $-t^3 + 3t^2 - 5t + 7$ 0	3 / ✓ 1 / ✓
	8_{10}^a $t^3 - 3t^2 + 6t - 7$ $-t^5 + 4t^4 - 11t^3 + 16t^2 - 21t + 20$	3 / ✗ 2 / ✗		8_{11}^a $-2t^2 + 7t - 9$ $5t^3 - 24t^2 + 39t - 44$	2 / ✗ 1 / ✗
	8_{12}^a $t^2 - 7t + 13$ 0	2 / ✗ 2 / ✓		8_{13}^a $2t^2 - 7t + 11$ $-t^3 + 4t^2 - 14t + 20$	2 / ✗ 1 / ✗
	8_{14}^a $-2t^2 + 8t - 11$ $5t^3 - 28t^2 + 57t - 68$	2 / ✗ 1 / ✗		8_{15}^a $3t^2 - 8t + 11$ $21t^3 - 64t^2 + 120t - 140$	2 / ✗ 2 / ✗
	8_{16}^a $t^3 - 4t^2 + 8t - 9$ $t^5 - 6t^4 + 17t^3 - 28t^2 + 35t - 36$	3 / ✗ 2 / ✗		8_{17}^a $-t^3 + 4t^2 - 8t + 11$ 0	3 / ✗ 1 / ✓
	8_{18}^a $-t^3 + 5t^2 - 10t + 13$ 0	3 / ✗ 2 / ✓		8_{19}^a $t^3 - t^2 + 1$ $-3t^5 - 4t^2 - 3t$	3 / ✗ 3 / ✗
	8_{20}^a $t^2 - 2t + 3$ $4t - 4$	2 / ✓ 1 / ✗		8_{21}^a $-t^2 + 4t - 5$ $t^3 - 8t^2 + 16t - 20$	2 / ✗ 1 / ✗
	9_1^a $t^4 - t^3 + t^2 - t + 1$ $4t^7 + 7t^5 + 9t^3 + 10t$	4 / ✗ 4 / ✗		9_2^a $4t - 7$ $30t - 40$	1 / ✗ 1 / ✗
	9_3^a $2t^3 - 3t^2 + 3t - 3$ $-13t^5 + 12t^4 - 25t^3 + 20t^2 - 32t + 24$	3 / ✗ 3 / ✗		9_4^a $3t^2 - 5t + 5$ $23t^3 - 28t^2 + 46t - 44$	2 / ✗ 2 / ✗
	9_5^a $6t - 11$ $100 - 65t$	1 / ✗ 2 / ✗		9_6^a $2t^3 - 4t^2 + 5t - 5$ $13t^5 - 24t^4 + 45t^3 - 52t^2 + 68t - 64$	3 / ✗ 3 / ✗
	9_7^a $3t^2 - 7t + 9$ $23t^3 - 56t^2 + 99t - 108$	2 / ✗ 2 / ✗		9_8^a $-2t^2 + 8t - 11$ $3t^3 - 16t^2 + 29t - 28$	2 / ✗ 2 / ✗
	9_9^a $2t^3 - 4t^2 + 6t - 7$ $13t^5 - 24t^4 + 55t^3 - 72t^2 + 98t - 96$	3 / ✗ 3 / ✗		9_{10}^a $4t^2 - 8t + 9$ $-40t^3 + 72t^2 - 114t + 120$	2 / ✗ 2,3 / ✗
	9_{11}^a $-t^3 + 5t^2 - 7t + 7$ $-2t^5 + 16t^4 - 41t^3 + 52t^2 - 66t + 64$	3 / ✗ 2 / ✗		9_{12}^a $-2t^2 + 9t - 13$ $5t^3 - 36t^2 + 84t - 100$	2 / ✗ 1 / ✗
	9_{13}^a $4t^2 - 9t + 11$ $-40t^3 + 92t^2 - 154t + 168$	2 / ✗ 2,3 / ✗		9_{14}^a $2t^2 - 9t + 15$ $-t^3 + 8t^2 - 35t + 60$	2 / ✗ 1 / ✗
	9_{15}^a $-2t^2 + 10t - 15$ $-5t^3 + 40t^2 - 108t + 136$	2 / ✗ 2 / ✗		9_{16}^a $2t^3 - 5t^2 + 8t - 9$ $-13t^5 + 36t^4 - 80t^3 + 120t^2 - 161t + 168$	3 / ✗ 3 / ✗
	9_{17}^a $t^3 - 5t^2 + 9t - 9$ $t^5 - 8t^4 + 23t^3 - 32t^2 + 28t - 24$	3 / ✗ 2 / ✗		9_{18}^a $4t^2 - 10t + 13$ $40t^3 - 108t^2 + 193t - 220$	2 / ✗ 2 / ✗
	9_{19}^a $2t^2 - 10t + 17$ $t^3 - 8t^2 + 20t - 24$	2 / ✗ 1 / ✗		9_{20}^a $-t^3 + 5t^2 - 9t + 11$ $2t^5 - 16t^4 + 47t^3 - 84t^2 + 117t - 124$	3 / ✗ 2 / ✗
	9_{21}^a $-2t^2 + 11t - 17$ $-5t^3 + 44t^2 - 127t + 164$	2 / ✗ 1 / ✗		9_{22}^a $t^3 - 5t^2 + 10t - 11$ $-t^5 + 8t^4 - 24t^3 + 38t^2 - 40t + 36$	3 / ✗ 1 / ✗
	9_{23}^a $4t^2 - 11t + 15$ $40t^3 - 128t^2 + 243t - 288$	2 / ✗ 2 / ✗		9_{24}^a $-t^3 + 5t^2 - 10t + 13$ $-4t^2 + 16t - 20$	3 / ✗ 1 / ✗
	9_{25}^a $-3t^2 + 12t - 17$ $12t^3 - 70t^2 + 153t - 188$	2 / ✗ 2 / ✗		9_{26}^a $t^3 - 5t^2 + 11t - 13$ $-t^5 + 8t^4 - 31t^3 + 64t^2 - 85t + 92$	3 / ✗ 1 / ✗
	9_{27}^a $-t^3 + 5t^2 - 11t + 15$ $t^5 - 8t^4 + 24t^3 - 32t^2 - 32t$	3 / ✓ 1 / ✗		9_{28}^a $t^3 - 5t^2 + 12t - 15$ $t^5 - 8t^4 + 30t^3 - 68t^2 + 105t - 120$	3 / ✗ 1 / ✗
	9_{29}^a $t^3 - 5t^2 + 12t - 15$ $t^5 - 8t^4 + 26t^3 - 48t^2 + 59t - 56$	3 / ✗ 2 / ✗		9_{30}^a $-t^3 + 5t^2 - 12t + 17$ $2t^3 - 10t^2 + 25t - 32$	3 / ✗ 1 / ✗

diagram	n_k^l Alexander a_+ P -essence e_+	genus / ribbon unknotting number / amphicheiral	diagram	n_k^l Alexander a_+ P -essence e_+	genus / ribbon unknotting number / amphicheiral
	$9a_{31} \quad t^3 - 5t^2 + 13t - 17$ $t^5 - 8t^4 + 33t^3 - 80t^2 + 132t - 152$	3 / ✗ 2 / ✗		$9a_{32} \quad t^3 - 6t^2 + 14t - 17$ $-t^5 + 10t^4 - 42t^3 + 94t^2 - 133t + 148$	3 / ✗ 2 / ✗
	$9a_{33} \quad -t^3 + 6t^2 - 14t + 19$ $t^3 - 10t^2 + 30t - 40$	3 / ✗ 1 / ✗		$9a_{34} \quad -t^3 + 6t^2 - 16t + 23$ $3t^3 - 18t^2 + 43t - 56$	3 / ✗ 1 / ✗
	$9a_{35} \quad 7t - 13$ $90t - 144$	1 / ✗ 2,3 / ✗		$9a_{36} \quad -t^3 + 5t^2 - 8t + 9$ $-2t^5 + 16t^4 - 44t^3 + 66t^2 - 87t + 88$	3 / ✗ 2 / ✗
	$9a_{37} \quad 2t^2 - 11t + 19$ $t^3 - 8t^2 + 22t - 28$	2 / ✗ 2 / ✗		$9a_{38} \quad 5t^2 - 14t + 19$ $62t^3 - 204t^2 + 382t - 452$	2 / ✗ 2,3 / ✗
	$9a_{39} \quad -3t^2 + 14t - 21$ $-12t^3 + 84t^2 - 210t + 268$	2 / ✗ 1 / ✗		$9a_{40} \quad t^3 - 7t^2 + 18t - 23$ $t^5 - 12t^4 + 57t^3 - 144t^2 + 229t - 264$	3 / ✗ 2 / ✗
	$9a_{41} \quad 3t^2 - 12t + 19$ $3t^3 - 20t^2 + 70t - 108$	2 / ✓ 2 / ✗		$9a_{42} \quad -t^2 + 2t - 1$ $-t^3 + 2t^2 + t - 4$	2 / ✗ 1 / ✗
	$9a_{43} \quad -t^3 + 3t^2 - 2t + 1$ $-2t^5 + 8t^4 - 7t^3 + 2t^2 - 5t + 4$	3 / ✗ 2 / ✗		$9a_{44} \quad t^2 - 4t + 7$ $-2t^2 + 9t - 12$	2 / ✗ 1 / ✗
	$9a_{45} \quad -t^2 + 6t - 9$ $t^3 - 14t^2 + 47t - 60$	2 / ✗ 1 / ✗		$9a_{46} \quad 5 - 2t$ $3t - 12$	1 / ✓ 2 / ✗
	$9a_{47} \quad t^3 - 4t^2 + 6t - 5$ $-t^5 + 6t^4 - 15t^3 + 16t^2 - 10t + 12$	3 / ✗ 2 / ✗		$9a_{48} \quad -t^2 + 7t - 11$ $-t^3 + 12t^2 - 42t + 52$	2 / ✗ 2 / ✗
	$9a_{49} \quad 3t^2 - 6t + 7$ $-21t^3 + 38t^2 - 61t + 60$	2 / ✗ 3 / ✗		$10a_1 \quad 9 - 4t$ $14t - 40$	1 / ✗ 1 / ✗
	$10a_2 \quad -t^4 + 3t^3 - 3t^2 + 3t - 3$ $3t^7 - 12t^6 + 16t^5 - 20t^4 + 24t^3 - 24t^2 + 27t - 24$	4 / ✗ 3 / ✗		$10a_3 \quad 13 - 6t$ $11t - 28$	1 / ✓ 2 / ✗
	$10a_4 \quad -3t^2 + 7t - 7$ $4t^3 - 8t^2 + t + 8$	2 / ✗ 2 / ✗		$10a_5 \quad t^4 - 3t^3 + 5t^2 - 5t + 5$ $-2t^7 + 8t^6 - 20t^5 + 28t^4 - 36t^3 + 36t^2 - 39t + 36$	4 / ✗ 2 / ✗
	$10a_6 \quad -2t^3 + 6t^2 - 7t + 7$ $9t^5 - 36t^4 + 56t^3 - 72t^2 + 81t - 84$	3 / ✗ 3 / ✗		$10a_7 \quad -3t^2 + 11t - 15$ $14t^3 - 72t^2 + 135t - 160$	2 / ✗ 1 / ✗
	$10a_8 \quad -2t^3 + 5t^2 - 5t + 5$ $7t^5 - 20t^4 + 23t^3 - 28t^2 + 26t - 24$	3 / ✗ 2 / ✗		$10a_9 \quad -t^4 + 3t^3 - 5t^2 + 7t - 7$ $-t^7 + 4t^6 - 10t^5 + 20t^4 - 25t^3 + 28t^2 - 28t + 28$	4 / ✗ 1 / ✗
	$10a_{10} \quad 3t^2 - 11t + 17$ $-5t^3 + 24t^2 - 71t + 100$	2 / ✗ 1 / ✗		$10a_{11} \quad -4t^2 + 11t - 13$ $16t^3 - 52t^2 + 68t - 72$	2 / ✗ 2,3 / ✗
	$10a_{12} \quad 2t^3 - 6t^2 + 10t - 11$ $-5t^5 + 20t^4 - 50t^3 + 72t^2 - 89t + 92$	3 / ✗ 2 / ✗		$10a_{13} \quad 2t^2 - 13t + 23$ $t^3 - 12t^2 + 51t - 84$	2 / ✗ 2 / ✗
	$10a_{14} \quad -2t^3 + 8t^2 - 12t + 13$ $9t^5 - 52t^4 + 119t^3 - 180t^2 + 225t - 236$	3 / ✗ 2 / ✗		$10a_{15} \quad 2t^3 - 6t^2 + 9t - 9$ $-3t^5 + 12t^4 - 24t^3 + 24t^2 - 17t + 12$	3 / ✗ 2 / ✗
	$10a_{16} \quad -4t^2 + 12t - 15$ $-16t^3 + 56t^2 - 76t + 80$	2 / ✗ 2 / ✗		$10a_{17} \quad t^4 - 3t^3 + 5t^2 - 7t + 9$ 0	4 / ✗ 1 / ✓
	$10a_{18} \quad -4t^2 + 14t - 19$ $16t^3 - 68t^2 + 121t - 140$	2 / ✗ 1 / ✗		$10a_{19} \quad 2t^3 - 7t^2 + 11t - 11$ $3t^5 - 16t^4 + 35t^3 - 40t^2 + 30t - 24$	3 / ✗ 2 / ✗
	$10a_{20} \quad -3t^2 + 9t - 11$ $14t^3 - 56t^2 + 88t - 104$	2 / ✗ 2 / ✗		$10a_{21} \quad -2t^3 + 7t^2 - 9t + 9$ $9t^5 - 44t^4 + 80t^3 - 104t^2 + 121t - 124$	3 / ✗ 2 / ✗
	$10a_{22} \quad -2t^3 + 6t^2 - 10t + 13$ $-t^5 + 4t^4 - 10t^3 + 24t^2 - 37t + 44$	3 / ✓ 2 / ✗		$10a_{23} \quad 2t^3 - 7t^2 + 13t - 15$ $-5t^5 + 24t^4 - 67t^3 + 108t^2 - 137t + 144$	3 / ✗ 1 / ✗
	$10a_{24} \quad -4t^2 + 14t - 19$ $24t^3 - 116t^2 + 221t - 268$	2 / ✗ 2 / ✗		$10a_{25} \quad -2t^3 + 8t^2 - 14t + 17$ $9t^5 - 52t^4 + 131t^3 - 232t^2 + 314t - 344$	3 / ✗ 2 / ✗
	$10a_{26} \quad -2t^3 + 7t^2 - 13t + 17$ $-t^5 + 4t^4 - 10t^3 + 28t^2 - 49t + 60$	3 / ✗ 1 / ✗		$10a_{27} \quad 2t^3 - 8t^2 + 16t - 19$ $5t^5 - 28t^4 + 87t^3 - 164t^2 + 229t - 252$	3 / ✗ 1 / ✗
	$10a_{28} \quad 4t^2 - 13t + 19$ $-8t^3 + 36t^2 - 100t + 136$	2 / ✗ 2 / ✗		$10a_{29} \quad t^3 - 7t^2 + 15t - 17$ $t^5 - 12t^4 + 52t^3 - 104t^2 + 124t - 128$	3 / ✗ 2 / ✗
	$10a_{30} \quad -4t^2 + 17t - 25$ $24t^3 - 148t^2 + 345t - 440$	2 / ✗ 1 / ✗		$10a_{31} \quad 4t^2 - 14t + 21$ $-4t^2 + 9t - 12$	2 / ✗ 1 / ✗
	$10a_{32} \quad -2t^3 + 8t^2 - 15t + 19$ $t^5 - 4t^4 + 13t^3 - 40t^2 + 78t - 96$	3 / ✗ 1 / ✗		$10a_{33} \quad 4t^2 - 16t + 25$ 0	2 / ✗ 1 / ✓
	$10a_{34} \quad 3t^2 - 9t + 13$ $-5t^3 + 20t^2 - 52t + 68$	2 / ✗ 2 / ✗		$10a_{35} \quad 2t^2 - 12t + 21$ $-t^3 + 12t^2 - 47t + 76$	2 / ✓ 2 / ✗
	$10a_{36} \quad -3t^2 + 13t - 19$ $14t^3 - 88t^2 + 208t - 264$	2 / ✗ 2 / ✗		$10a_{37} \quad 4t^2 - 13t + 19$ 0	2 / ✗ 2 / ✓
	$10a_{38} \quad -4t^2 + 15t - 21$ $24t^3 - 128t^2 + 270t - 336$	2 / ✗ 2 / ✗		$10a_{39} \quad -2t^3 + 8t^2 - 13t + 15$ $9t^5 - 52t^4 + 125t^3 - 204t^2 + 263t - 280$	3 / ✗ 2 / ✗
	$10a_{40} \quad 2t^3 - 8t^2 + 17t - 21$ $-5t^5 + 28t^4 - 89t^3 + 176t^2 - 258t + 288$	3 / ✗ 2 / ✗		$10a_{41} \quad t^3 - 7t^2 + 17t - 21$ $t^5 - 12t^4 + 54t^3 - 120t^2 + 157t - 164$	3 / ✗ 2 / ✗
	$10a_{42} \quad -t^3 + 7t^2 - 19t + 27$ $2t^3 - 8t^2 + 11t - 12$	3 / ✓ 1 / ✗		$10a_{43} \quad -t^3 + 7t^2 - 17t + 23$ 0	3 / ✗ 2 / ✓

diagram	n_k^l Alexander a_+ P -essence e_+	genus / ribbon unknotting number / amphicheiral	diagram	n_k^l Alexander a_+ P -essence e_+	genus / ribbon unknotting number / amphicheiral
	$t^3 - 7t^2 + 19t - 25$ $t^5 - 12t^4 + 56t^3 - 140t^2 + 220t - 248$	3 / ✗ 1 / ✗		$-t^3 + 7t^2 - 21t + 31$ 0	3 / ✗ 2 / ✓
	$-t^4 + 3t^3 - 4t^2 + 5t - 5$ $-3t^7 + 12t^6 - 21t^5 + 34t^4 - 43t^3 + 52t^2 - 55t + 56$	4 / ✗ 3 / ✗		$t^4 - 3t^3 + 6t^2 - 7t + 7$ $-2t^7 + 8t^6 - 23t^5 + 38t^4 - 56t^3 + 60t^2 - 68t + 64$	4 / ✗ 2, 3 / ✗
	$t^4 - 3t^3 + 6t^2 - 9t + 11$ $t^5 - 2t^4 + 2t^3 - 3t + 4$	4 / ✓ 2 / ✗		$3t^3 - 8t^2 + 12t - 13$ $30t^5 - 94t^4 + 196t^3 - 292t^2 + 372t - 392$	3 / ✗ 3 / ✗
	$-2t^3 + 7t^2 - 11t + 13$ $-9t^5 + 44t^4 - 94t^3 + 150t^2 - 186t + 200$	3 / ✗ 2 / ✗		$2t^3 - 7t^2 + 15t - 19$ $-5t^5 + 24t^4 - 73t^3 + 134t^2 - 194t + 212$	3 / ✗ 2, 3 / ✗
	$2t^3 - 7t^2 + 13t - 15$ $-3t^5 + 16t^4 - 37t^3 + 50t^2 - 49t + 44$	3 / ✗ 2 / ✗		$6t^2 - 18t + 25$ $93t^3 - 346t^2 + 680t - 828$	2 / ✗ 2, 3 / ✗
	$2t^3 - 6t^2 + 10t - 11$ $-3t^5 + 12t^4 - 24t^3 + 26t^2 - 21t + 16$	3 / ✗ 2, 3 / ✗		$5t^2 - 15t + 21$ $66t^3 - 246t^2 + 488t - 596$	2 / ✗ 2 / ✗
	$-2t^3 + 8t^2 - 14t + 17$ $-9t^5 + 52t^4 - 133t^3 + 234t^2 - 312t + 340$	3 / ✗ 2 / ✗		$2t^3 - 8t^2 + 18t - 23$ $-5t^5 + 28t^4 - 93t^3 + 194t^2 - 300t + 340$	3 / ✗ 2 / ✗
	$3t^2 - 16t + 27$ $3t^3 - 28t^2 + 94t - 140$	2 / ✗ 2 / ✗		$t^3 - 7t^2 + 18t - 23$ $-t^5 + 12t^4 - 55t^3 + 128t^2 - 181t + 196$	3 / ✗ 1 / ✗
	$-t^3 + 7t^2 - 20t + 29$ $5t^3 - 40t^2 + 122t - 176$	3 / ✗ 1 / ✗		$-2t^3 + 5t^2 - 6t + 7$ $-7t^5 + 20t^4 - 27t^3 + 36t^2 - 35t + 36$	3 / ✗ 2, 3 / ✗
	$t^4 - 3t^3 + 6t^2 - 8t + 9$ $-2t^7 + 8t^6 - 23t^5 + 40t^4 - 63t^3 + 76t^2 - 89t + 88$	4 / ✗ 2 / ✗		$5t^2 - 14t + 19$ $66t^3 - 220t^2 + 416t - 496$	2 / ✗ 2 / ✗
	$-t^4 + 3t^3 - 6t^2 + 10t - 11$ $-t^7 + 4t^6 - 11t^5 + 24t^4 - 37t^3 + 52t^2 - 60t + 64$	4 / ✗ 2 / ✗		$2t^3 - 7t^2 + 14t - 17$ $-5t^5 + 24t^4 - 71t^3 + 124t^2 - 169t + 180$	3 / ✗ 2 / ✗
	$3t^3 - 9t^2 + 16t - 19$ $30t^5 - 112t^4 + 279t^3 - 480t^2 + 662t - 724$	3 / ✗ 3 / ✗		$-4t^2 + 16t - 23$ $24t^3 - 140t^2 + 312t - 392$	2 / ✗ 2 / ✗
	$4t^2 - 14t + 21$ $8t^3 - 40t^2 + 117t - 164$	2 / ✗ 2 / ✗		$t^3 - 7t^2 + 21t - 29$ $-t^5 + 12t^4 - 68t^3 + 212t^2 - 397t + 476$	3 / ✗ 2 / ✗
	$t^3 - 7t^2 + 16t - 19$ $-t^5 + 12t^4 - 53t^3 + 114t^2 - 146t + 152$	3 / ✗ 2 / ✗		$-t^3 + 7t^2 - 18t + 25$ $t^3 - 2t^2 - t + 4$	3 / ✗ 1 / ✗
	$-2t^3 + 9t^2 - 16t + 19$ $-9t^5 + 60t^4 - 167t^3 + 298t^2 - 410t + 448$	3 / ✗ 2 / ✗		$t^3 - 7t^2 + 20t - 27$ $t^5 - 12t^4 + 65t^3 - 194t^2 + 350t - 416$	3 / ✗ 1 / ✗
	$-4t^2 + 16t - 23$ $24t^3 - 136t^2 + 290t - 360$	2 / ✗ 2 / ✗		$-t^3 + 7t^2 - 19t + 27$ $-4t^3 + 36t^2 - 117t + 172$	3 / ✓ 2 / ✗
	$-2t^3 + 7t^2 - 12t + 15$ $-9t^5 + 44t^4 - 104t^3 + 184t^2 - 245t + 272$	3 / ✗ 2, 3 / ✗		$2t^3 - 7t^2 + 14t - 17$ $-5t^5 + 24t^4 - 71t^3 + 132t^2 - 189t + 208$	3 / ✗ 2, 3 / ✗
	$-t^3 + 7t^2 - 16t + 21$ $2t^5 - 24t^4 + 105t^3 - 244t^2 + 390t - 448$	3 / ✗ 2 / ✗		$t^4 - 3t^3 + 7t^2 - 12t + 15$ 0	4 / ✗ 2, 3 / ✓
	$3t^3 - 9t^2 + 15t - 17$ $30t^5 - 112t^4 + 260t^3 - 426t^2 + 568t - 616$	3 / ✗ 3 / ✗		$-t^3 + 8t^2 - 20t + 27$ 0	3 / ✗ 2 / ✓
	$-t^4 + 4t^3 - 8t^2 + 12t - 13$ $t^7 - 6t^6 + 19t^5 - 42t^4 + 64t^3 - 78t^2 + 84t - 84$	4 / ✗ 1 / ✗		$2t^3 - 9t^2 + 19t - 23$ $-5t^5 + 34t^4 - 110t^3 + 214t^2 - 301t + 332$	3 / ✗ 2 / ✗
	$2t^3 - 9t^2 + 20t - 25$ $-5t^5 + 34t^4 - 116t^3 + 246t^2 - 373t + 424$	3 / ✗ 1 / ✗		$t^4 - 4t^3 + 8t^2 - 10t + 11$ $2t^7 - 12t^6 + 36t^5 - 68t^4 + 101t^3 - 124t^2 + 138t - 140$	4 / ✗ 2 / ✗
	$-2t^3 + 9t^2 - 19t + 25$ $-t^5 + 6t^4 - 21t^3 + 58t^2 - 105t + 128$	3 / ✗ 2 / ✗		$-2t^3 + 9t^2 - 18t + 23$ $-t^5 + 6t^4 - 23t^3 + 66t^2 - 125t + 152$	3 / ✓ 2 / ✗
	$-t^3 + 8t^2 - 24t + 35$ 0	3 / ✗ 1 / ✓		$t^3 - 8t^2 + 24t - 33$ $t^5 - 14t^4 + 83t^3 - 264t^2 + 495t - 596$	3 / ✗ 2 / ✗
	$-2t^3 + 8t^2 - 17t + 23$ $-t^5 + 6t^4 - 21t^3 + 54t^2 - 93t + 112$	3 / ✗ 2 / ✗		$t^4 - 4t^3 + 9t^2 - 14t + 17$ $t^5 - 2t^4 + 2t^3 - 3t + 4$	4 / ✗ 1 / ✗
	$-2t^3 + 10t^2 - 20t + 25$ $-9t^5 + 68t^4 - 216t^3 + 428t^2 - 622t + 696$	3 / ✗ 2 / ✗		$2t^3 - 8t^2 + 15t - 17$ $3t^5 - 18t^4 + 43t^3 - 58t^2 + 55t - 48$	3 / ✗ 2 / ✗
	$-t^4 + 4t^3 - 9t^2 + 14t - 15$ $-t^7 + 6t^6 - 20t^5 + 46t^4 - 76t^3 + 102t^2 - 115t + 120$	4 / ✗ 2 / ✗		$2t^3 - 9t^2 + 21t - 27$ $-5t^5 + 32t^4 - 114t^3 + 248t^2 - 384t + 436$	3 / ✗ 1 / ✗
	$-t^3 + 7t^2 - 22t + 33$ $-7t^3 + 50t^2 - 147t + 212$	3 / ✗ 2 / ✗		$-5t^2 + 22t - 33$ $-37t^3 + 242t^2 - 603t + 788$	2 / ✗ 2 / ✗
	$-2t^3 + 9t^2 - 18t + 23$ $9t^5 - 60t^4 + 177t^3 - 348t^2 + 501t - 564$	3 / ✗ 2 / ✗		$t^4 - 4t^3 + 10t^2 - 16t + 19$ 0	4 / ✓ 2 / ✓
	$t^4 - 4t^3 + 9t^2 - 12t + 13$ $2t^7 - 12t^6 + 39t^5 - 80t^4 + 128t^3 - 164t^2 + 192t - 196$	4 / ✗ 2, 3 / ✗		$7t^2 - 21t + 29$ $-129t^3 + 480t^2 - 942t + 1148$	2 / ✗ 2, 3 / ✗
	$-2t^3 + 8t^2 - 16t + 21$ $-t^5 + 6t^4 - 19t^3 + 50t^2 - 89t + 108$	3 / ✗ 1 / ✗		$2t^3 - 8t^2 + 17t - 21$ $5t^5 - 30t^4 + 93t^3 - 178t^2 + 254t - 280$	3 / ✗ 3 / ✗
	$t^4 - 4t^3 + 9t^2 - 15t + 19$ $t^5 - 2t^4 + 2t^3 - 3t + 4$	4 / ✗ 1 / ✗		$t^3 - 8t^2 + 22t - 29$ $-t^5 + 14t^4 - 71t^3 + 184t^2 - 292t + 332$	3 / ✗ 2 / ✗

diagram	n_k^l Alexander a_+ <i>P-essence</i> e_+	genus / ribbon unknotting number / amphicheiral	diagram	n_k^l Alexander a_+ <i>P-essence</i> e_+	genus / ribbon unknotting number / amphicheiral
	10_{106}^0 $-t^4 + 4t^3 - 9t^2 + 15t - 17$ $-t^7 + 6t^6 - 20t^5 + 48t^4 - 82t^3 + 114t^2 - 134t + 140$	4 / ✗ 2 / ✗		10_{107}^0 $-t^3 + 8t^2 - 22t + 31$ $2t^3 - 8t^2 + 13t - 16$	3 / ✗ 1 / ✗
	10_{108}^0 $2t^3 - 8t^2 + 14t - 15$ $-3t^5 + 18t^4 - 41t^3 + 50t^2 - 40t + 32$	3 / ✗ 2 / ✗		10_{109}^0 $t^4 - 4t^3 + 10t^2 - 17t + 21$ 0	4 / ✗ 2 / ✓
	10_{110}^0 $t^3 - 8t^2 + 20t - 25$ $t^5 - 14t^4 + 69t^3 - 160t^2 + 219t - 236$	3 / ✗ 2 / ✗		10_{111}^0 $-2t^3 + 9t^2 - 17t + 21$ $-9t^5 + 60t^4 - 171t^3 + 316t^2 - 436t + 480$	3 / ✗ 2 / ✗
	10_{112}^0 $-t^4 + 5t^3 - 11t^2 + 17t - 19$ $t^7 - 8t^6 + 29t^5 - 68t^4 + 115t^3 - 152t^2 + 175t - 180$	4 / ✗ 2 / ✗		10_{113}^0 $2t^3 - 11t^2 + 26t - 33$ $-5t^5 + 42t^4 - 167t^3 + 394t^2 - 623t + 720$	3 / ✗ 1 / ✗
	10_{114}^0 $-2t^3 + 10t^2 - 21t + 27$ $t^5 - 8t^4 + 30t^3 - 78t^2 + 140t - 168$	3 / ✗ 1 / ✗		10_{115}^0 $-t^3 + 9t^2 - 26t + 37$ 0	3 / ✗ 2 / ✓
	10_{116}^0 $-t^4 + 5t^3 - 12t^2 + 19t - 21$ $t^7 - 8t^6 + 30t^5 - 74t^4 + 132t^3 - 184t^2 + 217t - 228$	4 / ✗ 2 / ✗		10_{117}^0 $2t^3 - 10t^2 + 24t - 31$ $-5t^5 + 38t^4 - 144t^3 + 330t^2 - 522t + 600$	3 / ✗ 2 / ✗
	10_{118}^0 $t^4 - 5t^3 + 12t^2 - 19t + 23$ 0	4 / ✗ 1 / ✓		10_{119}^0 $-2t^3 + 10t^2 - 23t + 31$ $-t^5 + 6t^4 - 26t^3 + 86t^2 - 175t + 220$	3 / ✗ 1 / ✗
	10_{120}^0 $8t^2 - 26t + 37$ $166t^3 - 692t^2 + 1433t - 1788$	2 / ✗ 2, 3 / ✗		10_{121}^0 $2t^3 - 11t^2 + 27t - 35$ $5t^5 - 42t^4 + 167t^3 - 396t^2 + 634t - 732$	3 / ✗ 2 / ✗
	10_{122}^0 $-2t^3 + 11t^2 - 24t + 31$ $-t^5 + 8t^4 - 34t^3 + 104t^2 - 211t + 264$	3 / ✗ 2 / ✗		10_{123}^0 $t^4 - 6t^3 + 15t^2 - 24t + 29$ 0	4 / ✓ 2 / ✓
	10_{124}^0 $t^4 - t^3 + t - 1$ $-4t^7 - 6t^4 - 4t^2 - 6t$	4 / ✗ 4 / ✗		10_{125}^0 $t^3 - 2t^2 + 2t - 1$ $-t^5 + 2t^4 - 2t^3 + 3t - 4$	3 / ✗ 2 / ✗
	10_{126}^0 $t^3 - 2t^2 + 4t - 5$ $t^5 - 2t^4 + 10t^3 - 12t^2 + 22t - 20$	3 / ✗ 2 / ✗		10_{127}^0 $-t^3 + 4t^2 - 6t + 7$ $2t^5 - 14t^4 + 32t^3 - 52t^2 + 67t - 72$	3 / ✗ 2 / ✗
	10_{128}^0 $2t^3 - 3t^2 + t + 1$ $-13t^5 + 12t^4 - 3t^3 - 10t^2 - 9t + 12$	3 / ✗ 3 / ✗		10_{129}^0 $2t^2 - 6t + 9$ $-t^3 - 2t^2 + 14t - 20$	2 / ✓ 1 / ✗
	10_{130}^0 $2t^2 - 4t + 5$ $t^3 - 2t^2 + 19t - 24$	2 / ✗ 2 / ✗		10_{131}^0 $-2t^2 + 8t - 11$ $5t^3 - 38t^2 + 87t - 112$	2 / ✗ 1 / ✗
	10_{132}^0 $t^2 - t + 1$ $2t^2 + 5t - 4$	2 / ✗ 1 / ✗		10_{133}^0 $-t^2 + 5t - 7$ $t^3 - 14t^2 + 37t - 48$	2 / ✗ 1 / ✗
	10_{134}^0 $2t^3 - 4t^2 + 4t - 3$ $-13t^5 + 24t^4 - 33t^3 + 30t^2 - 41t + 40$	3 / ✗ 3 / ✗		10_{135}^0 $3t^2 - 9t + 13$ $t^3 - 6t^2 + 18t - 24$	2 / ✗ 2 / ✗
	10_{136}^0 $-t^2 + 4t - 5$ $-t^3 + 4t^2 - 2t - 4$	2 / ✗ 1 / ✗		10_{137}^0 $t^2 - 6t + 11$ $-4t^2 + 24t - 44$	2 / ✓ 1 / ✗
	10_{138}^0 $t^3 - 5t^2 + 8t - 7$ $-t^5 + 8t^4 - 22t^3 + 24t^2 - 11t + 8$	3 / ✗ 2 / ✗		10_{139}^0 $t^4 - t^3 + 2t - 3$ $-4t^7 - 12t^4 + 5t^3 - 4t^2 - 16t + 12$	4 / ✗ 4 / ✗
	10_{140}^0 $t^2 - 2t + 3$ $8t - 8$	2 / ✓ 2 / ✗		10_{141}^0 $-t^3 + 3t^2 - 4t + 5$ $t^3 - 8t^2 + 16t - 20$	3 / ✗ 1 / ✗
	10_{142}^0 $2t^3 - 3t^2 + 2t - 1$ $-13t^5 + 12t^4 - 13t^3 + 4t^2 - 17t + 12$	3 / ✗ 3 / ✗		10_{143}^0 $t^3 - 3t^2 + 6t - 7$ $t^5 - 4t^4 + 15t^3 - 28t^2 + 45t - 48$	3 / ✗ 1 / ✗
	10_{144}^0 $-3t^2 + 10t - 13$ $10t^3 - 44t^2 + 80t - 96$	2 / ✗ 2 / ✗		10_{145}^0 $t^2 + t - 3$ $2t^3 + 8t^2 + 6t - 8$	2 / ✗ 2 / ✗
	10_{146}^0 $2t^2 - 8t + 13$ $t^3 - 8t^2 + 21t - 28$	2 / ✗ 1 / ✗		10_{147}^0 $-2t^2 + 7t - 9$ $-3t^3 + 12t^2 - 15t + 12$	2 / ✗ 1 / ✗
	10_{148}^0 $t^3 - 3t^2 + 7t - 9$ $t^5 - 4t^4 + 18t^3 - 36t^2 + 62t - 68$	3 / ✗ 2 / ✗		10_{149}^0 $-t^3 + 5t^2 - 9t + 11$ $2t^5 - 18t^4 + 55t^3 - 104t^2 + 149t - 164$	3 / ✗ 2 / ✗
	10_{150}^0 $-t^3 + 4t^2 - 6t + 7$ $-2t^5 + 12t^4 - 26t^3 + 38t^2 - 45t + 44$	3 / ✗ 2 / ✗		10_{151}^0 $t^3 - 4t^2 + 10t - 13$ $-t^5 + 6t^4 - 21t^3 + 42t^2 - 66t + 72$	3 / ✗ 2 / ✗
	10_{152}^0 $t^4 - t^3 - t^2 + 4t - 5$ $4t^7 - 7t^5 + 18t^4 - 7t^3 - 12t^2 + 45t - 52$	4 / ✗ 4 / ✗		10_{153}^0 $t^3 - t^2 - t + 3$ $t^5 - 2t^4 + t^3 + 2t^2 - t$	3 / ✓ 2 / ✗
	10_{154}^0 $t^3 - 4t + 7$ $-3t^5 - 6t^4 + 13t^3 - 47t + 68$	3 / ✗ 3 / ✗		10_{155}^0 $-t^3 + 3t^2 - 5t + 7$ $-2t^3 + 12t^2 - 22t + 28$	3 / ✓ 2 / ✗
	10_{156}^0 $t^3 - 4t^2 + 8t - 9$ $t^5 - 6t^4 + 19t^3 - 30t^2 + 33t - 32$	3 / ✗ 1 / ✗		10_{157}^0 $-t^3 + 6t^2 - 11t + 13$ $-2t^5 + 22t^4 - 78t^3 + 148t^2 - 218t + 240$	3 / ✗ 2 / ✗
	10_{158}^0 $-t^3 + 4t^2 - 10t + 15$ $2t^2 - 7t + 12$	3 / ✗ 2 / ✗		10_{159}^0 $t^3 - 4t^2 + 9t - 11$ $t^5 - 6t^4 + 26t^3 - 60t^2 + 98t - 112$	3 / ✗ 1 / ✗
	10_{160}^0 $-t^3 + 4t^2 - 4t + 3$ $-2t^5 + 12t^4 - 20t^3 + 14t^2 - 16t + 12$	3 / ✗ 2 / ✗		10_{161}^0 $t^3 - 2t + 3$ $3t^5 + 6t^4 - 3t^3 + 4t^2 + 14t - 12$	3 / ✗ 3 / ✗
	10_{162}^0 $-3t^2 + 9t - 11$ $10t^3 - 38t^2 + 58t - 68$	2 / ✗ 2 / ✗		10_{163}^0 $t^3 - 5t^2 + 12t - 15$ $-t^5 + 8t^4 - 30t^3 + 62t^2 - 89t + 96$	3 / ✗ 1, 2 / ✗
	10_{164}^0 $3t^2 - 11t + 17$ $t^3 - 10t^2 + 29t - 40$	2 / ✗ 1 / ✗		10_{165}^0 $-2t^2 + 10t - 15$ $-5t^3 + 50t^2 - 146t + 196$	2 / ✗ 2 / ✗