Pensieve header: Implementing ρ_1, and also ρ_d.

exec

```
nb2tex$TeXFileName = "Rho1.tex";
```

pdf

Preliminaries

pdf

This is Rho.nb of http://drorbn.net/la22/ap.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\UCLA-221104"];
```

pdf

In[ ]:=
```
Once[<< KnotTheory`; << Rot.m];
```

pdf

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
```

pdf

```
Loading Rot.m from http://drorbn.net/la22/ap to compute rotation numbers.
```

pdf

The Program

pdf

In[ ]:=
```
R1[s_, i_, j_] := s (gji (gj*,j + gj,j* - gij) - gii (gj,j* - 1) - 1 / 2);
Z[K_] := Module[{Cs, φ, n, A, s, i, j, k, Δ, G, ρ1},
   {Cs, φ} = Rot[K]; n = Length[Cs];
   A = IdentityMatrix[2 n + 1];
   Cases[Cs, {s_, i_, j_} :> (A[[{i, j}, {i + 1, j + 1}]] += ({{-T^s, T^s - 1}, {0, -1}}))];
   Δ = T^(-Total[φ] - Total[Cs[[All, 1]]]) / 2 Det[A];
   G = Inverse[A];
   ρ1 = Sum[R1 @@ Cs[[k]], {k, 1, n}] - Sum[φ[[k]] (gkk - 1 / 2), {k, 1, 2 n}];
   Factor@{Δ, Δ^2 ρ1 /. α_^+ :> α + 1 /. gα_,β_ :> G[[α, β]]}];
```

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The First Few Knots

pdf

In[ ]:=
```
TableForm[Table[Join[{K[[1]]_K[[2]]}, Z[K]], {K, AllKnots[{3, 6}]}], TableAlignments → Center]
```

pdf

KnotTheory: Loading precomputed data in PD4Knots`.

Out[ ]//TableForm=

pdf

3_1	$\frac{1-T+T^2}{T}$	$\frac{(-1+T)^2\,(1+T^2)}{T^2}$
4_1	$-\frac{1-3\,T+T^2}{T}$	0
5_1	$\frac{1-T+T^2-T^3+T^4}{T^2}$	$\frac{(-1+T)^2\,(1+T^2)\,(2+T^2+2\,T^4)}{T^4}$
5_2	$\frac{2-3\,T+2\,T^2}{T}$	$\frac{(-1+T)^2\,(5-4\,T+5\,T^2)}{T^2}$
6_1	$-\frac{(-2+T)\,(-1+2\,T)}{T}$	$\frac{(-1+T)^2\,(1-4\,T+T^2)}{T^2}$
6_2	$-\frac{1-3\,T+3\,T^2-3\,T^3+T^4}{T^2}$	$\frac{(-1+T)^2\,(1-4\,T+4\,T^2-4\,T^3+4\,T^4-4\,T^5+T^6)}{T^4}$
6_3	$\frac{1-3\,T+5\,T^2-3\,T^3+T^4}{T^2}$	0

tex

```
\def\nbpdfText#1{\vskip -3mm\[ \includegraphics[width=0.4\linewidth]{#1}\quad p=1-T^s \]}
```

pdf

tex

```
\def\nbpdfText#1{\vskip 1mm\par\noindent\includegraphics{#1}}
```

tex

```
\needspace{2in}
```

pdf

Fast!

tex

```
\[ \resizebox{\linewidth}{!}{\import{../Waco-2203/}{GST48-Marked.pdf_t}} \]
```

pdf

In[•]:=
```
Timing@
 Z[GST48 = EPD[X14,1, X2,29, X3,40, X43,4, X26,5, X6,95, X96,7, X13,8, X9,28, X10,41, X42,11, X27,12, X30,15,
    X16,61, X17,72, X18,83, X19,34, X89,20, X21,92, X79,22, X68,23, X57,24, X25,56, X62,31, X73,32,
    X84,33, X50,35, X36,81, X37,70, X38,59, X39,54, X44,55, X58,45, X69,46, X80,47, X48,91, X90,49,
    X51,82, X52,71, X53,60, X63,74, X64,85, X76,65, X87,66, X67,94, X75,86, X88,77, X78,93]]
```

Out[•]=

pdf

$$\left\{170.313,\ \left\{-\frac{\left(-1+2\,T-T^2-T^3+2\,T^4-T^5+T^8\right)\,\left(-1+T^3-2\,T^4+T^5+T^6-2\,T^7+T^8\right)}{T^8},\right.\right.$$
$$\frac{1}{T^{16}}\,(-1+T)^2\,\left(5-18\,T+33\,T^2-32\,T^3+2\,T^4+42\,T^5-62\,T^6-8\,T^7+166\,T^8-242\,T^9+108\,T^{10}+\right.$$
$$\left.\left.\left.132\,T^{11}-226\,T^{12}+148\,T^{13}-11\,T^{14}-36\,T^{15}-11\,T^{16}+148\,T^{17}-226\,T^{18}+132\,T^{19}+108\,T^{20}-\right.\right.\right.$$
$$\left.\left.\left.242\,T^{21}+166\,T^{22}-8\,T^{23}-62\,T^{24}+42\,T^{25}+2\,T^{26}-32\,T^{27}+33\,T^{28}-18\,T^{29}+5\,T^{30}\right)\right\}\right\}$$

pdf

Strong!

pdf

```
{NumberOfKnots[{3, 12}],
 Length@Union@Table[Z[K], {K, AllKnots[{3, 12}]}],
 Length@Union@Table[{HOMFLYPT[K], Kh[K]}, {K, AllKnots[{3, 12}]}]}
```

Out[•]=

pdf

```
{2977, 2882, 2785}
```

In[•]:=
```
2977 - {2882, 2785}
```

Out[•]=

```
{95, 192}
```

tex

So the pair (Δ,ρ_1) attains 2,882 distinct values on the 2,977 prime knots with up to 12 crossings (a deficit of 95), whereas the pair (HOMFLYPT, Khovanov Homology) attains only 2,785 distinct values on the same knots (a deficit of 192).

tex

\def\nbpdfText#1{\vskip 1mm\par\noindent\includegraphics[width=\linewidth]{#1}}

pdf

tex

\def\nbpdfText#1{\vskip 1mm\par\noindent\includegraphics{#1}}

Invariance under R3

exec

```
nb2tex$TeXFileName = "Invariance.tex";
```

pdf

```
In[•]:= δ_i_,j_ := If[i === j, 1, 0];
        gRules_s_,i_,j_ := {g_iβ_ :> δ_iβ + T^s g_i⁺,β + (1 - T^s) g_j⁺,β, g_jβ_ :> δ_jβ + g_j⁺,β,
          g_α_,i :> T^-s (g_α,i⁺ - δ_α,i⁺), g_α_,j :> g_α,j⁺ - (1 - T^s) g_αi - δ_α,j⁺}
```

Proof of Reidemeister 3:

pdf

```
In[•]:= lhs = R1[1, j, k] + R1[1, i, k⁺] + R1[1, i⁺, j⁺] //. gRules1,j,k ⋃ gRules1,i,k⁺ ⋃ gRules1,i⁺,j⁺;
        rhs = R1[1, i, j] + R1[1, i⁺, k] + R1[1, j⁺, k⁺] //. gRules1,i,j ⋃ gRules1,i⁺,k ⋃ gRules1,j⁺,k⁺;
        Simplify[lhs == rhs]
```

Out[•]=
pdf

True

tex

Next comes Reid1, where we use results from an earlier example:

```
In[•]:= {{1, T^-1, 1}, {0, T^-1, 1}, {0, 0, 1}} // Inverse // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & T & -T \\ 0 & 0 & 1 \end{pmatrix}$$

pdf

```
In[•]:= R1[1, 2, 1] - 1 (g22 - 1 / 2) /. g_α_,β_ :> {{1, T^-1, 1}, {0, T^-1, 1}, {0, 0, 1}}[[α, β]]
```

Out[•]=
pdf

$$\frac{1}{T^2} - \frac{1}{T} - \frac{-1 + \frac{1}{T}}{T}$$

tex

Invariance under the other moves is proven similarly.

exec

```
nb2tex$TeXFileName = "Rhod.tex";
nb2tex$PDFWidth = 4.2 / 0.7;
```

On to ρ_d!

tex

{\bf\red Implementation.} Data, then program (with output using the \text{Conway} variable $z=\sqrt{T}-1/\sqrt{T}$), and then a demo. See {\tt Rho.nb} of \web{ap}.
\def\nbpdfInput#1{\vskip 1mm\par\noindent\includegraphics[scale=0.7]{#1}}
\def\nbpdfOutput#1{\vskip 1mm\par\noindent\includegraphics[scale=0.7]{#1}}

pdf

```
In[•]:= V@γ1,φ_[k_] = φ (1 / 2 - p̄k x̄k); V@γ2,φ_[k_] = -φ^2 p̄k x̄k / 2; V@γ3,φ_[k_] := -φ^3 p̄k x̄k / 6
```

pdf

In[ ]:=
```
V@r_{1,s_}[i_, j_] :=
 s (-1 + 2 p_i x_i - 2 p_j x_i + (-1 + T^s) p_i p_j x_i^2 + (1 - T^s) p_j^2 x_i^2 - 2 p_i p_j x_i x_j + 2 p_j^2 x_i x_j) / 2
```

pdf

In[ ]:=
```
V@r_{2,1}[i_, j_] := (-6 p_i x_i + 6 p_j x_i - 3 (-1 + 3 T) p_i p_j x_i^2 + 3 (-1 + 3 T) p_j^2 x_i^2 + 4 (-1 + T) p_i^2 p_j x_i^3 -
     2 (-1 + T) (5 + T) p_i p_j^2 x_i^3 + 2 (-1 + T) (3 + T) p_j^3 x_i^3 + 18 p_i p_j x_i x_j - 18 p_j^2 x_i x_j -
     6 p_i^2 p_j x_i^2 x_j + 6 (2 + T) p_i p_j^2 x_i^2 x_j - 6 (1 + T) p_j^3 x_i^2 x_j - 6 p_i p_j^2 x_i x_j^2 + 6 p_j^3 x_i x_j^2) / 12
```

pdf

In[ ]:=
```
V@r_{2,-1}[i_, j_] :=
 (-6 T^2 p_i x_i + 6 T^2 p_j x_i + 3 (-3 + T) T p_i p_j x_i^2 - 3 (-3 + T) T p_j^2 x_i^2 - 4 (-1 + T) T p_i^2 p_j x_i^3 + 2 (-1 + T)
      (1 + 5 T) p_i p_j^2 x_i^3 - 2 (-1 + T) (1 + 3 T) p_j^3 x_i^3 + 18 T^2 p_i p_j x_i x_j - 18 T^2 p_j^2 x_i x_j - 6 T^2 p_i^2 p_j x_i^2 x_j +
    6 T (1 + 2 T) p_i p_j^2 x_i^2 x_j - 6 T (1 + T) p_j^3 x_i^2 x_j - 6 T^2 p_i p_j^2 x_i x_j^2 + 6 T^2 p_j^3 x_i x_j^2) / (12 T^2)
```

pdf

In[ ]:=
```
V@r_{3,1}[i_, j_] :=
 (4 p_i x_i - 4 p_j x_i + 2 (5 + 7 T) p_i p_j x_i^2 - 2 (5 + 7 T) p_j^2 x_i^2 - 4 (-5 + 6 T) p_i^2 p_j x_i^3 + 4 (-16 + 17 T + 2 T^2)
      p_i p_j^2 x_i^3 - 4 (-11 + 11 T + 2 T^2) p_j^3 x_i^3 + 3 (-1 + T) p_i^3 p_j x_i^4 - 3 (-1 + T) (4 + 3 T) p_i^2 p_j^2 x_i^4 +
    (-1 + T) (13 + 22 T + T^2) p_i p_j^3 x_i^4 - (-1 + T) (4 + 13 T + T^2) p_j^4 x_i^4 - 28 p_i p_j x_i x_j + 28 p_j^2 x_i x_j +
    36 p_i^2 p_j x_i^2 x_j - 12 (9 + 2 T) p_i p_j^2 x_i^2 x_j + 24 (3 + T) p_j^3 x_i^2 x_j - 4 p_i^3 p_j x_i^3 x_j + 28 T p_i^2 p_j^2 x_i^3 x_j -
    4 (-6 + 17 T + T^2) p_i p_j^3 x_i^3 x_j + 4 (-5 + 10 T + T^2) p_j^4 x_i^3 x_j + 24 p_i p_j^2 x_i x_j^2 - 24 p_j^3 x_i x_j^2 -
    24 p_i^2 p_j^2 x_i^2 x_j^2 + 6 (10 + T) p_i p_j^3 x_i^2 x_j^2 - 6 (6 + T) p_j^4 x_i^2 x_j^2 - 4 p_i p_j^3 x_i x_j^3 + 4 p_j^4 x_i x_j^3) / 24
```

pdf

In[ ]:=
```
V@r_{3,-1}[i_, j_] :=
 (-4 T^3 p_i x_i + 4 T^3 p_j x_i - 2 T^2 (7 + 5 T) p_i p_j x_i^2 + 2 T^2 (7 + 5 T) p_j^2 x_i^2 - 4 T^2 (-6 + 5 T) p_i^2 p_j x_i^3 +
    4 T (-2 - 17 T + 16 T^2) p_i p_j^2 x_i^3 - 4 T (-2 - 11 T + 11 T^2) p_j^3 x_i^3 + 3 (-1 + T) T^2 p_i^3 p_j x_i^4 -
    3 (-1 + T) T (3 + 4 T) p_i^2 p_j^2 x_i^4 + (-1 + T) (1 + 22 T + 13 T^2) p_i p_j^3 x_i^4 -
    (-1 + T) (1 + 13 T + 4 T^2) p_j^4 x_i^4 + 28 T^3 p_i p_j x_i x_j - 28 T^3 p_j^2 x_i x_j - 36 T^3 p_i^2 p_j x_i^2 x_j +
    12 T^2 (2 + 9 T) p_i p_j^2 x_i^2 x_j - 24 T^2 (1 + 3 T) p_j^3 x_i^2 x_j + 4 T^3 p_i^3 p_j x_i^3 x_j -
    28 T^2 p_i^2 p_j^2 x_i^3 x_j - 4 T (-1 - 17 T + 6 T^2) p_i p_j^3 x_i^3 x_j + 4 T (-1 - 10 T + 5 T^2) p_j^4 x_i^3 x_j -
    24 T^3 p_i p_j^2 x_i x_j^2 + 24 T^3 p_j^3 x_i x_j^2 + 24 T^3 p_i^2 p_j^2 x_i^2 x_j^2 - 6 T^2 (1 + 10 T) p_i p_j^3 x_i^2 x_j^2 +
    6 T^2 (1 + 6 T) p_j^4 x_i^2 x_j^2 + 4 T^3 p_i p_j^3 x_i x_j^3 - 4 T^3 p_j^4 x_i x_j^3) / (24 T^3)
```

pdf

In[ ]:=
```
{p^*, x^*, p̄^*, x̄^*} = {π, ξ, π̄, ξ̄};  (z_i__)^* := (z^*)_i;
Zip_{}[ℰ_] := ℰ;
Zip_{z_,zs___}[ℰ_] := (Collect[ℰ // Zip_{zs}, z] /. f_. z^(d_.) :> (D[f, {z^*, d}])) /. z^* → 0
```

pdf

```
In[ ]:= gPair[fs_, w_] := gPair[fs, w] = Collect[Zip_{Join@@Table[{p_α, p̄_α, x_α, x̄_α}, {α, w}]}[(Times @@ (V /@ fs)) Exp[Sum[g_{α,β} (π_α + π̄_α) (ξ_β + ξ̄_β), {α, w}, {β, w}] - Sum[ξ̄_α π_α, {α, w}]]], g__, Factor]
```

pdf

```
In[ ]:= T2z[p_] := Module[{q = Expand[p], n, c},
  If[q === 0, 0, c = Coefficient[q, T, n = Exponent[q, T]];
   c z^(2 n) + T2z[q - c (T^(1/2) - T^(-1/2))^(2 n)]]];
```

pdf

```
In[ ]:= Z_d_[K_] := Module[{Cs, φ, n, A, s, i, j, k, Δ, G, d1, Z1, Z2, Z3},
  {Cs, φ} = Rot[K]; n = Length[Cs]; A = IdentityMatrix[2 n + 1];
  Cases[Cs, {s_, i_, j_} :> (A[[{i, j}, {i + 1, j + 1}]] += ({{-T^s, T^s - 1}, {0, -1}}))];
  {Δ, G} = Factor@{T^((-Total[φ] - Total[Cs[[All, 1]]])/2) Det@A, Inverse@A};
  Z1 = Exp[Total[Cases[Cs, {s_, i_, j_} :> Sum[ϵ^d1 r_{d1,s}[i, j], {d1, d}]]] +
     Sum[ϵ^d1 γ_{d1,φ[[k]]}[k], {k, 2 n}, {d1, d}] /. γ_,0[_] → 0];
  Z2 = Expand[F[{}, {}] × Normal@Series[Z1, {ϵ, 0, d}]] //. F[fs_, {es___}] ×
     (f : (r | γ)_ps__[is__])^p_. :> F[Join[fs, Table[f, p]], DeleteDuplicates@{es, is}];
  Z3 = Expand[Z2 /. F[fs_, es_] :> Expand[gPair[
       Replace[fs, Thread[es → Range@Length@es], {2}], Length@es
     ] /. g_α_,β_ :> G[[es[[α]], es[[β]]]]]];
  Collect[{Δ, Z3 /. ϵ^p_. → p! Δ^(2 p) ϵ^p}, ϵ, T2z]];
```

```
In[ ]:= Z_3[Knot[3, 1]] // Timing
```

KnotTheory: Loading precomputed data in PD4Knots`.

Out[ ]=

$$\left\{49.9844,\ \left\{1+z^2,\ 1+\left(2z^2+z^4\right)\epsilon+\left(2-4z^2+3z^4+4z^6+z^8\right)\epsilon^2+\left(-12+74z^2-27z^4-20z^6+8z^8+6z^{10}+z^{12}\right)\epsilon^3\right\}\right\}$$

```
In[ ]:= Z_3[Knot[3, 1]] // Timing
```

Out[ ]=

$$\left\{1.26563,\ \left\{1+z^2,\ 1+\left(2z^2+z^4\right)\epsilon+\left(2-4z^2+3z^4+4z^6+z^8\right)\epsilon^2+\left(-12+74z^2-27z^4-20z^6+8z^8+6z^{10}+z^{12}\right)\epsilon^3\right\}\right\}$$

Demos

exec

```
nb2tex$PDFWidth = 8 / 0.75;
```

tex

\end{multicols}
\def\nbpdfInput#1{\vskip 1mm\par\noindent\includegraphics[scale=0.75]{#1}}

\def\nbpdfOutput#1{\vskip 1mm\par\noindent\includegraphics[scale=0.75]{#1}}

In[•]:=
```
GST48 = EPD[X[14,1], Xbar[2,29], X[3,40], X[43,4], Xbar[26,5], X[6,95], X[96,7], X[13,8], Xbar[9,28], X[10,41], X[42,11], Xbar[27,12],
   X[30,15], Xbar[16,61], Xbar[17,72], Xbar[18,83], X[19,34], Xbar[89,20], Xbar[21,92], Xbar[79,22], Xbar[68,23], Xbar[57,24], Xbar[25,56], X[62,31],
   X[73,32], X[84,33], Xbar[50,35], X[36,81], X[37,70], X[38,59], Xbar[39,54], X[44,55], X[58,45], X[69,46], X[80,47], X[48,91],
   X[90,49], X[51,82], X[52,71], X[53,60], Xbar[63,74], Xbar[64,85], Xbar[76,65], Xbar[87,66], Xbar[67,94], Xbar[75,86], Xbar[88,77], Xbar[78,93]];
Z2[GST48] // Timing
Z2[GST48] // Timing
```

Out[•]=
```
{564.578, {1 - 4 z^2 - 61 z^4 - 207 z^6 - 296 z^8 - 210 z^10 - 77 z^12 - 14 z^14 - z^16,
  1 + (38 z^2 + 255 z^4 + 1696 z^6 + 16 281 z^8 + 86 952 z^10 + 259 994 z^12 + 487 372 z^14 + 615 066 z^16 +
       543 148 z^18 + 341 714 z^20 + 153 722 z^22 + 48 983 z^24 + 10 776 z^26 + 1554 z^28 + 132 z^30 + 5 z^32) ∈ +
   (-8 - 484 z^2 + 9709 z^4 + 165 952 z^6 + 1 590 491 z^8 + 16 256 508 z^10 + 115 341 797 z^12 + 432 685 748 z^14 +
       395 838 354 z^16 - 4 017 557 792 z^18 - 23 300 064 167 z^20 - 70 082 264 972 z^22 - 142 572 271 191 z^24 -
       209 475 503 700 z^26 - 221 616 295 209 z^28 - 151 502 648 428 z^30 - 23 700 199 243 z^32 +
       99 462 146 328 z^34 + 164 920 463 074 z^36 + 162 550 825 432 z^38 + 119 164 552 296 z^40 +
       69 153 062 608 z^42 + 32 547 596 611 z^44 + 12 541 195 448 z^46 + 3 961 384 155 z^48 + 1 021 219 696 z^50 +
       212 773 106 z^52 + 35 264 208 z^54 + 4 537 548 z^56 + 436 600 z^58 + 29 536 z^60 + 1252 z^62 + 25 z^64) ∈^2}}
```

Out[•]=
```
{598.109, {1 - 4 z^2 - 61 z^4 - 207 z^6 - 296 z^8 - 210 z^10 - 77 z^12 - 14 z^14 - z^16,
  1 + (38 z^2 + 255 z^4 + 1696 z^6 + 16 281 z^8 + 86 952 z^10 + 259 994 z^12 + 487 372 z^14 + 615 066 z^16 +
       543 148 z^18 + 341 714 z^20 + 153 722 z^22 + 48 983 z^24 + 10 776 z^26 + 1554 z^28 + 132 z^30 + 5 z^32) ∈ +
   (-8 - 484 z^2 + 9709 z^4 + 165 952 z^6 + 1 590 491 z^8 + 16 256 508 z^10 + 115 341 797 z^12 + 432 685 748 z^14 +
       395 838 354 z^16 - 4 017 557 792 z^18 - 23 300 064 167 z^20 - 70 082 264 972 z^22 - 142 572 271 191 z^24 -
       209 475 503 700 z^26 - 221 616 295 209 z^28 - 151 502 648 428 z^30 - 23 700 199 243 z^32 +
       99 462 146 328 z^34 + 164 920 463 074 z^36 + 162 550 825 432 z^38 + 119 164 552 296 z^40 +
       69 153 062 608 z^42 + 32 547 596 611 z^44 + 12 541 195 448 z^46 + 3 961 384 155 z^48 + 1 021 219 696 z^50 +
       212 773 106 z^52 + 35 264 208 z^54 + 4 537 548 z^56 + 436 600 z^58 + 29 536 z^60 + 1252 z^62 + 25 z^64) ∈^2}}
```

pdf
```
Z2[GST48] (* takes a few minutes *)
```

Out[•]=
pdf
```
{1 - 4 z^2 - 61 z^4 - 207 z^6 - 296 z^8 - 210 z^10 - 77 z^12 - 14 z^14 - z^16,
 1 + (38 z^2 + 255 z^4 + 1696 z^6 + 16 281 z^8 + 86 952 z^10 + 259 994 z^12 + 487 372 z^14 + 615 066 z^16 +
      543 148 z^18 + 341 714 z^20 + 153 722 z^22 + 48 983 z^24 + 10 776 z^26 + 1554 z^28 + 132 z^30 + 5 z^32) ∈ +
  (-8 - 484 z^2 + 9709 z^4 + 165 952 z^6 + 1 590 491 z^8 + 16 256 508 z^10 + 115 341 797 z^12 + 432 685 748 z^14 +
      395 838 354 z^16 - 4 017 557 792 z^18 - 23 300 064 167 z^20 - 70 082 264 972 z^22 - 142 572 271 191 z^24 -
      209 475 503 700 z^26 - 221 616 295 209 z^28 - 151 502 648 428 z^30 - 23 700 199 243 z^32 +
      99 462 146 328 z^34 + 164 920 463 074 z^36 + 162 550 825 432 z^38 + 119 164 552 296 z^40 +
      69 153 062 608 z^42 + 32 547 596 611 z^44 + 12 541 195 448 z^46 + 3 961 384 155 z^48 + 1 021 219 696 z^50 +
      212 773 106 z^52 + 35 264 208 z^54 + 4 537 548 z^56 + 436 600 z^58 + 29 536 z^60 + 1252 z^62 + 25 z^64) ∈^2}
```

In[•]:=
```
Table[Join[{K⟦1⟧_K⟦2⟧}, Z3[K]], {K, AllKnots[{3, 6}]}] // Timing
```

Out[•]=

$$\{256.063, \{\{3_1, 1+z^2, 1+\left(2z^2+z^4\right)\epsilon+\left(2-4z^2+3z^4+4z^6+z^8\right)\epsilon^2+\left(-12+74z^2-27z^4-20z^6+8z^8+6z^{10}+z^{12}\right)\epsilon^3\},$$
$$\{4_1, 1-z^2, 1+\left(-2+2z^4\right)\epsilon^2\}, \{5_1, 1+3z^2+z^4, 1+\left(10z^2+21z^4+12z^6+2z^8\right)\epsilon+$$
$$\left(6-28z^2+33z^4+364z^6+655z^8+536z^{10}+227z^{12}+48z^{14}+4z^{16}\right)\epsilon^2+$$
$$\left(-60+970z^2+645z^4-3380z^6-3280z^8+7470z^{10}+19475z^{12}+\right.$$
$$\left.20536z^{14}+12564z^{16}+4774z^{18}+1109z^{20}+144z^{22}+8z^{24}\right)\epsilon^3\},$$
$$\{5_2, 1+2z^2, 1+\left(6z^2+5z^4\right)\epsilon+\left(4-20z^2+43z^4+64z^6+26z^8\right)\epsilon^2+$$
$$\left(-36+498z^2-883z^4+100z^6+816z^8+556z^{10}+146z^{12}\right)\epsilon^3\},$$
$$\{6_1, 1-2z^2, 1+\left(-2z^2+z^4\right)\epsilon+\left(-4+4z^2+25z^4-8z^6+2z^8\right)\epsilon^2+$$
$$\left(12+154z^2-223z^4-608z^6+100z^8-52z^{10}+10z^{12}\right)\epsilon^3\},$$
$$\{6_2, 1-z^2-z^4, 1+\left(-2z^2-3z^4+2z^6+z^8\right)\epsilon+$$
$$\left(-2-4z^2+29z^4+28z^6+42z^8-8z^{10}-2z^{12}+4z^{14}+z^{16}\right)\epsilon^2+\left(12+166z^2+155z^4-\right.$$
$$\left.194z^6-2453z^8-1622z^{10}-1967z^{12}-258z^{14}+49z^{16}-30z^{18}+z^{20}+6z^{22}+z^{24}\right)\epsilon^3\},$$
$$\{6_3, 1+z^2+z^4, 1+\left(2+8z^2-16z^6-24z^8-16z^{10}-2z^{12}\right)\epsilon^2\}\}\}$$

In[•]:=
```
Table[Join[{K⟦1⟧_K⟦2⟧}, Z3[K]], {K, AllKnots[{3, 6}]}] // Timing
```

Out[•]=

$$\{143.641, \{\{3_1, 1+z^2, 1+\left(2z^2+z^4\right)\epsilon+\left(2-4z^2+3z^4+4z^6+z^8\right)\epsilon^2+\left(-12+74z^2-27z^4-20z^6+8z^8+6z^{10}+z^{12}\right)\epsilon^3\},$$
$$\{4_1, 1-z^2, 1+\left(-2+2z^4\right)\epsilon^2\}, \{5_1, 1+3z^2+z^4, 1+\left(10z^2+21z^4+12z^6+2z^8\right)\epsilon+$$
$$\left(6-28z^2+33z^4+364z^6+655z^8+536z^{10}+227z^{12}+48z^{14}+4z^{16}\right)\epsilon^2+$$
$$\left(-60+970z^2+645z^4-3380z^6-3280z^8+7470z^{10}+19475z^{12}+\right.$$
$$\left.20536z^{14}+12564z^{16}+4774z^{18}+1109z^{20}+144z^{22}+8z^{24}\right)\epsilon^3\},$$
$$\{5_2, 1+2z^2, 1+\left(6z^2+5z^4\right)\epsilon+\left(4-20z^2+43z^4+64z^6+26z^8\right)\epsilon^2+$$
$$\left(-36+498z^2-883z^4+100z^6+816z^8+556z^{10}+146z^{12}\right)\epsilon^3\},$$
$$\{6_1, 1-2z^2, 1+\left(-2z^2+z^4\right)\epsilon+\left(-4+4z^2+25z^4-8z^6+2z^8\right)\epsilon^2+$$
$$\left(12+154z^2-223z^4-608z^6+100z^8-52z^{10}+10z^{12}\right)\epsilon^3\},$$
$$\{6_2, 1-z^2-z^4, 1+\left(-2z^2-3z^4+2z^6+z^8\right)\epsilon+$$
$$\left(-2-4z^2+29z^4+28z^6+42z^8-8z^{10}-2z^{12}+4z^{14}+z^{16}\right)\epsilon^2+\left(12+166z^2+155z^4-\right.$$
$$\left.194z^6-2453z^8-1622z^{10}-1967z^{12}-258z^{14}+49z^{16}-30z^{18}+z^{20}+6z^{22}+z^{24}\right)\epsilon^3\},$$
$$\{6_3, 1+z^2+z^4, 1+\left(2+8z^2-16z^6-24z^8-16z^{10}-2z^{12}\right)\epsilon^2\}\}\}$$

tex

```
\def\nbpdfOutput#1{\vskip 1mm\par\noindent\includegraphics[width=\linewidth]{#1}}
```

pdf

```
TableForm[Table[Join[{K[[1]]_K[[2]]}, Z_3[K]], {K, AllKnots[{3, 6}]}], TableAlignments → Center]
(* takes a few minutes *)
```

pdf

KnotTheory: Loading precomputed data in PD4Knots`.

Out[ ]//TableForm=
pdf

3_1	$1+z^2$	$1+\left(2z^2+z^4\right)\epsilon$
4_1	$1-z^2$	
5_1	$1+3z^2+z^4$	$1+\left(10z^2+21z^4+12z^6+2z^8\right)\epsilon+\left(6-28z^2+33z^4+364z^6+655z^8+536z^{10}+227$
5_2	$1+2z^2$	$1+\left(6z^2+5z^4\right)\epsilon+\left(4-2$
6_1	$1-2z^2$	$1+\left(-2z^2+z^4\right)\epsilon+\left(-$
6_2	$1-z^2-z^4$	$1+\left(-2z^2-3z^4+2z^6+z^8\right)\epsilon+\left(-2-4z^2+29z^4+28z^6+42z^8-8$
6_3	$1+z^2+z^4$	