Pensieve header: Examples for “Geography vs. Identity”. Most material is from TurboGassner.nb at pensieve://2019-11/.

P

In[∘]:= $\delta\ /\!:\ \delta_{i\_,j\_} := \mathrm{If}[i == j, 1, 0];$

In[∘]:= $\tau_{i\_,j\_}[\mathcal{E}\_] := \left(\mathcal{E}\ /.\ \left\{z\_{k\_} \mapsto z_{k/.\{i\to j, j\to i\}},\ z\_{l\_,k\_} \mapsto z_{l/.\{i\to j,j\to i\},k/.\{i\to j,j\to i\}}\right\}\right)$
$(*\ \text{Non-linear over } \mathbb{Q}\left(t_i^{\pm1}\right)!\ *)$

The Burau Representation

B

In[∘]:= $B_{i\_,j\_}[\mathcal{E}\_] := \mathcal{E}\ /.\ v_{k\_} \mapsto v_k + \delta_{k,j}\left(t-1\right)\left(v_j - v_i\right)\ //\ \mathrm{Expand}$

B

In[∘]:= $\left(\mathrm{bas3} = \{v_1, v_2, v_3\}\right)\ //\ B_{1,2}$

B

Out[∘]= $\{v_1,\ v_1 - t\, v_1 + t\, v_2,\ v_3\}$

B

In[∘]:= $\mathrm{bas3}\ //\ B_{1,2}\ //\ B_{1,3}\ //\ B_{2,3}$

B

Out[∘]= $\left\{v_1,\ v_1 - t\, v_1 + t\, v_2,\ v_1 - t\, v_1 + t\, v_2 - t^2 v_2 + t^2 v_3\right\}$

B

In[∘]:= $\mathrm{bas3}\ //\ B_{2,3}\ //\ B_{1,3}\ //\ B_{1,2}$

B

Out[∘]= $\left\{v_1,\ v_1 - t\, v_1 + t\, v_2,\ v_1 - t\, v_1 + t\, v_2 - t^2 v_2 + t^2 v_3\right\}$

The Gassner Representation

G

In[∘]:= $G_{i\_,j\_}[\mathcal{E}\_] := \mathcal{E}\ /.\ v_{k\_} \mapsto v_k + \delta_{k,j}\left(t_i-1\right)\left(v_j - v_i\right)\ //\ \mathrm{Expand}$

G

In[∘]:= $\left(\mathrm{bas3}\ //\ G_{1,2}\ //\ G_{1,3}\ //\ G_{2,3}\right) == \left(\mathrm{bas3}\ //\ G_{2,3}\ //\ G_{1,3}\ //\ G_{1,2}\right)$

G

Out[∘]= True

In[∘]:= $\{v_1, v_2, v_3\}\ //\ G_{1,2}\ //\ G_{1,3}$
$\{v_1, v_2, v_3\}\ //\ G_{1,3}\ //\ G_{1,2}$

Out[∘]= $\{v_1,\ v_1 - t_1 v_1 + t_1 v_2,\ v_1 - t_1 v_1 + t_1 v_3\}$

Out[∘]= $\{v_1,\ v_1 - t_1 v_1 + t_1 v_2,\ v_1 - t_1 v_1 + t_1 v_3\}$

In[∘]:= $\{v_1, v_2, v_3\}\ //\ G_{1,3}\ //\ G_{2,3}$
$\{v_1, v_2, v_3\}\ //\ G_{2,3}\ //\ G_{1,3}$

Out[∘]= $\{v_1,\ v_2,\ v_1 - t_1 v_1 + t_1 v_2 - t_1 t_2 v_2 + t_1 t_2 v_3\}$

Out[∘]= $\{v_1,\ v_2,\ t_2 v_1 - t_1 t_2 v_1 + v_2 - t_2 v_2 + t_1 t_2 v_3\}$

In[∘]:= $(\{v_1, v_2, v_3\} \mathbin{//} G_{1,2} \mathbin{//} \tau_{1,2} \mathbin{//} \tau_{2,3} \mathbin{//} \tau_{1,2}) - (\{v_1, v_2, v_3\} \mathbin{//} \tau_{2,3} \mathbin{//} \tau_{1,2} \mathbin{//} G_{2,3} \mathbin{//} \tau_{2,3})$

Out[∘]= {0, 0, 0}

In[∘]:= $(\{v_1, v_2, v_3\} \mathbin{//} G_{1,2} \mathbin{//} \tau_{2,3} \mathbin{//} \tau_{1,2}) - (\{v_1, v_2, v_3\} \mathbin{//} \tau_{2,3} \mathbin{//} \tau_{1,2} \mathbin{//} G_{2,3})$

Out[∘]= {0, 0, 0}

The Turbo-Gassner Representation

TG

In[∘]:=
$$\begin{aligned}
&TG_{i_,j_}[\xi_] := \xi \;/.\; \{\\
&\quad v_{k_} \mapsto v_k + \delta_{k,j}\left((t_i - 1)(v_j - v_i) + v_{i,j} - v_{i,i}\right) + \delta_{k,i}(u_j - u_i)\, u_i\, w_j,\\
&\quad v_{l_,k_} \mapsto v_{l,k} + (t_i - 1) \times\\
&\qquad \left(\delta_{k,j}(v_{l,j} - v_{l,i}) + (\delta_{l,i} - \delta_{l,j}\, t_i^{-1}\, t_j)\left(u_k + \delta_{k,j}(t_i - 1)(u_j - u_i)\right) u_i\, w_j\right),\\
&\quad u_{k_} \mapsto u_k + \delta_{k,j}(t_i - 1)(u_j - u_i),\\
&\quad w_{k_} \mapsto w_k + (\delta_{k,j} - \delta_{k,i})(t_i^{-1} - 1)\, w_j\} \mathbin{//} Expand
\end{aligned}$$

TG

In[∘]:=
$$\begin{aligned}
bas3 = \{&v_1, v_2, v_3, v_{1,1}, v_{1,2}, v_{1,3}, v_{2,1}, v_{2,2}, v_{2,3}, v_{3,1},\\
&v_{3,2}, v_{3,3}, u_1^2 w_1, u_1^2 w_2, u_1^2 w_3, u_1 u_2 w_1, u_1 u_2 w_2, u_1 u_2 w_3, u_1 u_3 w_1, u_1 u_3 w_2,\\
&u_1 u_3 w_3, u_2^2 w_1, u_2^2 w_2, u_2^2 w_3, u_2 u_3 w_1, u_2 u_3 w_2, u_2 u_3 w_3, u_3^2 w_1, u_3^2 w_2, u_3^2 w_3\};
\end{aligned}$$
$$(bas3 \mathbin{//} TG_{1,2} \mathbin{//} TG_{1,3} \mathbin{//} TG_{2,3}) == (bas3 \mathbin{//} TG_{2,3} \mathbin{//} TG_{1,3} \mathbin{//} TG_{1,2})$$

TG

Out[∘]= True

In[∘]:= $(bas3 \mathbin{//} TG_{1,2} \mathbin{//} \tau_{2,3} \mathbin{//} \tau_{1,2}) == (bas3 \mathbin{//} \tau_{2,3} \mathbin{//} \tau_{1,2} \mathbin{//} TG_{2,3})$

Out[∘]= True