Chord Diagrams, Knots, and Lie Algebras

Thanks for inviting me to the *Chord Diagrams Everywhere* session / Winter 2019 CMS meeting!
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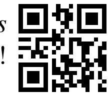
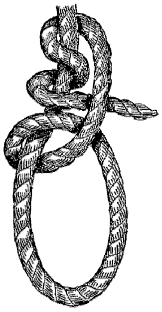
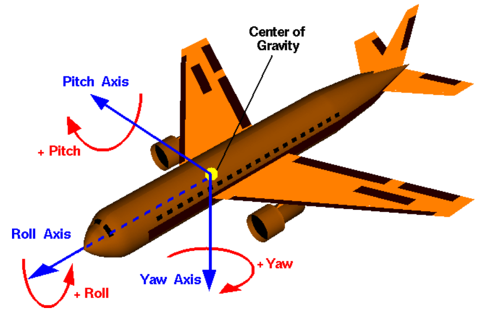


Abstract. This will be a service talk on ancient material — I will briefly describe how the exact same type of chord diagrams (and relations between them) occur in a natural way in both knot theory and in the theory of Lie algebras.

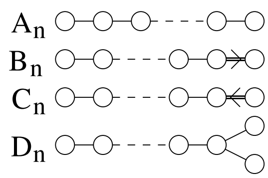
While preparing for this talk I realized that I've done it before, much better, within a book review. So here's that review! It has been modified from its original version: it had been formatted to fit this page, parts were highlighted, and commentary had been added in green italics.

[Book] *Introduction to Vassiliev Knot Invariants,* by S. Chmutov, S. Duzhin, and J. Mostovoy, Cambridge University Press, Cambridge UK, 2012, xvi+504 pp., hardback, \$70.00, ISBN 978-1-10702-083-2.





$$[X,Y]=Z \quad [Y,Z]=X \quad [Z,X]=Y \qquad Y=\frac{1}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$$



A knot and a Lie algebra, a list of knots and a list of Lie algebras, and an unusual conference of the symmetric and the knotted.

Merely ~~30~~ *36* years ago, if you had asked even the best informed mathematician about the relationship between knots and Lie algebras, she would have laughed, for there isn't and there can't be. Knots are flexible; Lie algebras are rigid. Knots are irregular; Lie algebras are symmetric. The list of knots is a lengthy mess; the collection of Lie algebras is well-organized. Knots are useful for sailors, scouts, and hangmen; Lie algebras for navigators, engineers, and high energy physicists. Knots are blue collar; Lie algebras are white. They are as similar as worms and crystals: both well-studied, but hardly ever together.

Then in the 1980s came Jones, and Witten, and Reshetikhin and Turaev [Jo, Wi, RT] and showed that if you really are the best informed, and you know your quantum field theory and conformal field theory and quantum groups, then you know that the two disjoint fields are in fact intricately related. This "quantum" approach remains the most powerful way to get computable knot invariants out of (certain) Lie algebras (and representations thereof). Yet shortly later, in the late 80s and early 90s, an alternative perspective arose, that of "finite-type" or "Vassiliev-Goussarov" invariants [Va1, Va2, Go1, Go2, BL, Ko1, Ko2, BN1], which made the surprising relationship between knots and Lie algebras appear simple and almost inevitable.

The reviewed [Book] is about that alternative perspective, the one reasonable sounding but not entirely trivial theorem that is crucially needed within it (the "Fundamental Theorem" or the "Kontsevich integral"), and the many threads that begin with that perspective. Let me start with a brief summary of the mathematics, and even before, an even briefer summary.

In briefest, a certain space $\mathcal{A}$ of chord diagrams is the dual to the dual of the space of knots, and at the same time, it is dual to Lie algebras.

The briefer summary is that in some combinatorial sense it is possible to "differentiate" knot invariants, and hence it makes sense to talk about "polynomials" on the space of knots — these are functions on the set of knots (namely, these are knot invariants) whose sufficiently high derivatives vanish. Such polynomials can be fairly conjectured to separate knots — elsewhere in math in lucky cases polynomials separate points, and in our case, specific computations are encouraging. Also, such polynomials are determined by their "coefficients", and each of these, by the one-side-easy "Fundamental Theorem", is a linear functional on some finite space of

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graphs modulo relations. These same graphs turn out to parameterize formulas that make sense in a wide class of Lie algebras, and the said relations match exactly with the relations in the definition of a Lie algebra — anti-symmetry and the Jacobi identity. Hence what is more or less dual to knots (invariants), is also, after passing to the coefficients, dual to certain graphs which are more or less dual to Lie algebras. QED, and on to the less brief summary[1].

Let V be an arbitrary invariant of oriented knots in oriented space with values in (say) $\mathbb{Q}$. Extend V to be an invariant of 1-singular knots, knots that have a single singularity that locally looks like a double point $\times$, using the formula

$$V(\times) = V(\nearrow\!\!\!\!\nwarrow) - V(\nwarrow\!\!\!\!\nearrow). \tag{1}$$

Further extend V to the set $\mathcal{K}^m$ of m-singular knots (knots with m such double points) by repeatedly using (1).

Definition 1. We say that V is of type m (or "Vassiliev of type m") if its extension $V|_{\mathcal{K}^{m+1}}$ to $(m+1)$-singular knots vanishes identically. We say that V is of finite type (or "Vassiliev") if it is of type m for some m.

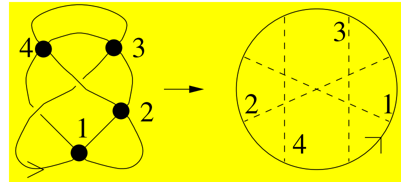
Repeated differences are similar to repeated derivatives and hence it is fair to think of the definition of $V|_{\mathcal{K}^m}$ as repeated differentiation. With this in mind, the above definition imitates the definition of polynomials of degree m. Hence finite type invariants can be thought of as "polynomials" on the space of knots[2]. It is known (see e.g. [Book]) that the class of finite type invariants is large and powerful. Yet the first question on finite type invariants remains unanswered:

Problem 2. *Honest polynomials are dense in the space of functions. Are finite type invariants dense within the space of all knot invariants? Do they separate knots?*

The top derivatives of a multi-variable polynomial form a system of constants that determine that polynomial up to polynomials of lower degree. Likewise the mth derivative[3] $V^{(m)} = V|_{\mathcal{K}^m} = V\left(\times \overset{m}{\cdots} \times\right)$ of a type m invariant V is a constant in the sense that it does not see the difference between overcrossings and undercrossings and so it is blind to 3D topology. Indeed

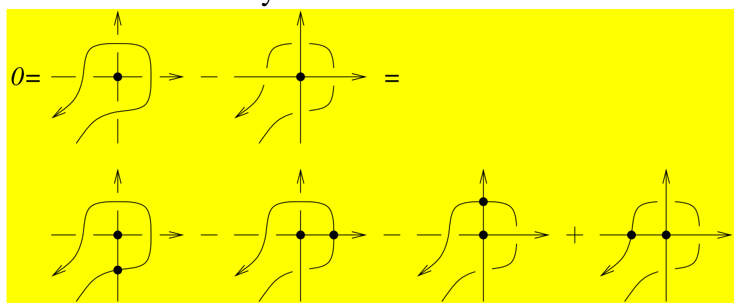
$$V\left(\times \overset{m}{\cdots} \times \nearrow\!\!\!\!\nwarrow\right) - V\left(\times \overset{m}{\cdots} \times \nwarrow\!\!\!\!\nearrow\right) = V\left(\times \overset{m+1}{\cdots} \times\right) = 0.$$

Also, clearly $V^{(m)}$ determines V up to invariants of lower type. Hence a primary tool in the study of finite type invariants is the study of the "top derivative" $V^{(m)}$, also known as "the weight system of V".



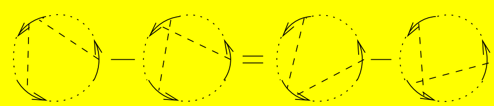
Blind to 3D topology, $V^{(m)}$ only sees the combinatorics of the circle that parameterizes an m-singular knot. On this circle there are m pairs of points that are pairwise identified in the image; standardly one indicates those by drawing a circle with m chords marked (an "m-chord diagram") as above. Let $\mathcal{D}_m$ denote the space of all formal linear combinations with rational coefficients of m-chord diagrams. Thus $V^{(m)}$ is a linear functional on $\mathcal{D}_m$.

I leave it for the reader to figure out or read in [Book, pp. 88] how the following figure easily implies the "4T" relations of the "easy side" of the theorem that follows:



Theorem 3. *(The Fundamental Theorem, details in* [Book]*).*

- *(Easy side) If V is a rational valued type m invariant then $V^{(m)}$ satisfies the "4T" relations shown above, and hence it descends to a linear functional on $\mathcal{A}_m := \mathcal{D}_m/4T$. If in addition $V^{(m)} \equiv 0$, then V is of type $m-1$.*



- *(Hard side, slightly misstated by avoiding "framings") For any linear functional W on $\mathcal{A}_m$ there is a rational valued type m invariant V so that $V^{(m)} = W$.*

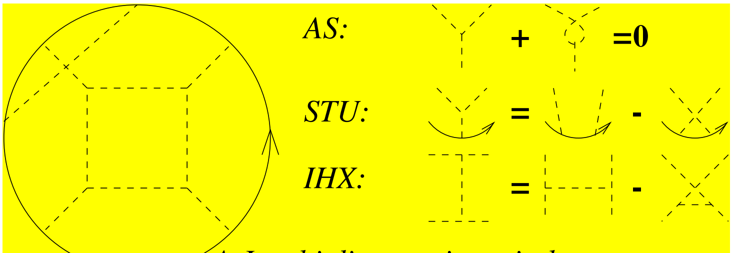
Thus to a large extent the study of finite type invariants is reduced to the finite (though super-exponential in m) algebraic study of $\mathcal{A}_m$.

Much of the richness of finite type invariants stems from their relationship with Lie algebras. Theorem 4 below suggests this relationship on an abstract level and Theorem 5 makes that relationship concrete.

[1] Partially self-plagiarized from [BN2].

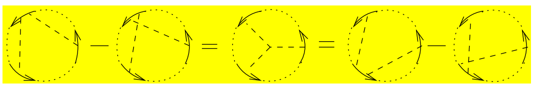
[2] Keep this apart from invariants of knots whose values are polynomials, such as the Alexander or the Jones polynomial. A posteriori related, these are a priori entirely different.

[3] As common in the knot theory literature, in the formulas that follow a picture such as $\times \overset{m}{\cdots} \times \nearrow\!\!\!\!\nwarrow$ indicates "some knot having m double points and a further (right-handed) crossing". Furthermore, when two such pictures appear within the same formula, it is to be understood that the parts of the knots (or diagrams) involved *outside* of the displayed pictures are to be taken as the same.



A Jacobi diagram in a circle

Theorem 4. [BN1] *The space $\mathcal{A}_m$ is isomorphic to the space $\mathcal{A}^t_m$ generated by "Jacobi diagrams in a circle" (chord diagrams that are also allowed to have oriented internal trivalent vertices) that have exactly 2m vertices, modulo the AS, STU and IHX relations. See the figure above.*

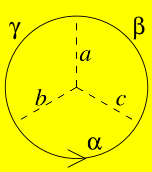


The key to the proof of Theorem 4 is the figure above, which shows that the $4T$ relation is a consequence of two STU relations. The rest is more or less an exercise in induction.

Thinking of internal trivalent vertices as graphical analogs of the Lie bracket, the AS relation becomes the anti-commutativity of the bracket, STU becomes the equation $[x, y] = xy - yx$ and IHX becomes the Jacobi identity. This analogy is made concrete within the following construction, originally due to Penrose [Pe] and to Cvitanović [Cv]. Given a finite dimensional metrized Lie algebra $\mathfrak{g}$ (e.g., any semi-simple Lie algebra) and a finite-dimensional representation $\rho : \mathfrak{g} \to \mathrm{End}(V)$ of $\mathfrak{g}$, choose an orthonormal basis[4] $\{X_a\}_{a=1}^{\dim\mathfrak{g}}$ of $\mathfrak{g}$ and some basis $\{v_\alpha\}_{\alpha=1}^{\dim V}$ of V, let f_{abc} and $r^\gamma_{\alpha\beta}$ be the "structure constants" defined by

$$f_{abc} := \langle [X_a, X_b], X_c\rangle \qquad \text{and} \qquad \rho(X_a)(v_\beta) = \sum_\gamma r^\gamma_{a\beta} v_\gamma.$$

Now given a Jacobi diagram D label its circle-arcs with Greek letters $\alpha, \beta, \ldots$, and its chords with Latin letters a, b, …, and map it to a sum as suggested by the following example:



$$\longrightarrow \sum_{a,b,c,\alpha,\beta,\gamma} f_{abc} r^\beta_{a\gamma} r^\gamma_{b\alpha} r^\alpha_{c\beta}$$

(internal vertices go to f's, circle-vertices to r's)

Theorem 5. *This construction is well defined, and the basic properties of Lie algebras imply that it respects the AS, STU, and IHX relations. Therefore it defines a linear functional $W_{\mathfrak{g},\rho} : \mathcal{A}_m \to \mathbb{Q}$, for any m.*

The last assertion along with Theorem 3 show that associated with any $\mathfrak{g}$, ρ and m there is a weight system and hence a knot invariant. Thus knots are indeed linked with Lie algebras.

The above is of course merely a sketch of the beginning of a long story. You can read the details, and some of the rest, in [Book].

[4] This requirement can easily be relaxed.

What I like about [Book]**.** Detailed, well thought out, and carefully written. Lots of pictures! Many excellent exercises! A complete discussion of "the algebra of chord diagrams". A nice discussion of the pairing of diagrams with Lie algebras, including examples aplenty. The discussion of the Kontsevich integral (meaning, the proof of the hard side of Theorem 3) is terrific — detailed and complete and full of pictures and examples, adding a great deal to the original sources. The subject of "associators" is huge and worthy of its own book(s); yet in as much as they are related to Vassiliev invariants, the discussion in [Book] is excellent. A great many further topics are touched — multiple ζ-values, the relationship of the Hopf link with the Duflo isomorphism, intersection graphs and other combinatorial aspects of chord diagrams, Rozansky's rationality conjecture, the Melvin-Morton conjecture, braids, n-equivalence, etc.

For all these, I'd certainly recommend [Book] to any newcomer to the subject of knot theory, starting with my own students.

However, some proofs other than that of Theorem 3 are repeated as they appear in original articles with only a superficial touch-up, or are omitted altogether, thus missing an opportunity to clarify some mysterious points. This includes Vogel's construction of a non-Lie-algebra weight system and the Goussarov-Polyak-Viro proof of the existence of "Gauss diagram formulas".

What I wish there was in the book, but there isn't. The relationship with Chern-Simons theory, Feynman diagrams, and configuration space integrals, culminating in an alternative (and more "3D") proof of the Fundamental Theorem. This is a major omission.

Why I hope there will be a continuation book, one day. There's much more to the story! There are finite type invariants of 3-manifolds, and of certain classes of 2-dimensional knots in $\mathbb{R}^4$, and of "virtual knots", and they each have their lovely yet non-obvious theories, and these theories link with each other and with other branches of Lie theory, algebra, topology, and quantum field theory. Volume 2 is sorely needed.

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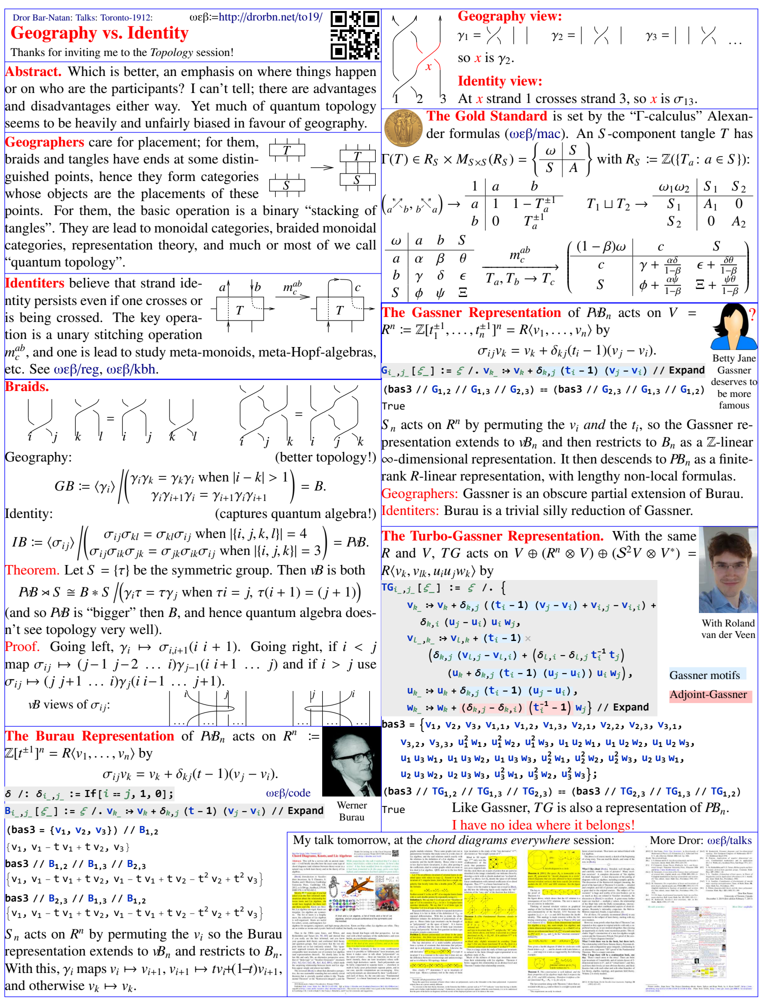
My talk yesterday:



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