Work in Progress!

A Poly-Time Knot Polynomial Via Solvable Approximation

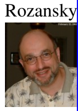
Abstract. I will construct the first poly-time-computable knot polynomial since Alexander's [Al, 1928] by using some new commutator-calculus techniques and a Lie algebra $\mathfrak{g}_1$ which is at the same time solvable and an approximation of the simple Lie algebra sl_2.

Expected! Finite-type invariants include all coefficients of all quantum knot polynomials (appropriately parametrized), and each is computable in poly-time. Yet

d	2	3	4	5	6	7	8	$\cdots$
known f.t. invts in $O(n^d)$	1	1	∞	3	4	8	11	$\cdots$

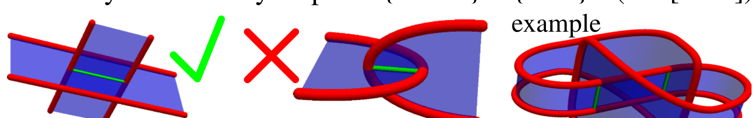
This is an unreasonable picture! So there ought to be further poly-time polynomial invariants.

Also. • The line above the Alexander line in the Melvin-Morton [MM, Ro] expansion of the coloured Jones polynomial. • The 2-loop contribution to the Kontsevich integral.


Rozansky

Paradise! Foremost reason: *OBVIOUSLY.* Cf. proving (incomputable A)=(incomputable B), or categorifying (incomputable C).

Secondary reason: may disprove {ribbon} = {slice}: (see [BN2])



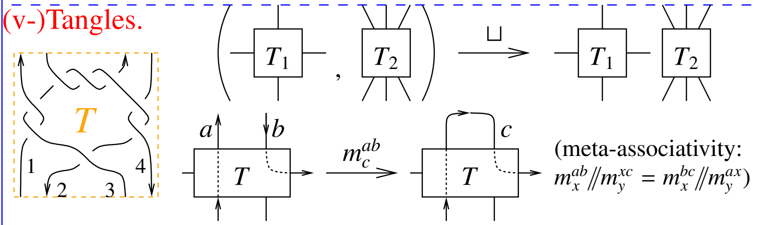
a ribbon singularity a clasp singularity

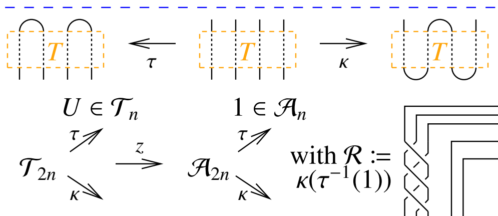
A bit about ribbon knots. A "ribbon knot" is a knot that can be presented as the boundary of a disk that has "ribbon singularities", but no "clasp singularities". A "slice knot" is a knot in $S^3 = \partial B^4$ which is the boundary of a non-singular disk in B^4. Every ribbon knots is clearly slice, yet,

Conjecture. Some slice knots are not ribbon.

Fox-Milnor. The Alexander polynomial of a ribbon knot is always of the form $A(t) = f(t)f(1/t)$. (also for slice)

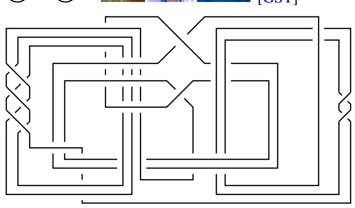
(v-)Tangles.

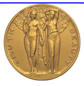





Gompf, Scharlemann, Thompson [GST]

Faster is better, leaner is meaner!





The Gold Standard is set by the "Γ-calculus" Alexander formulas [BNS, BN1]. An S-component tangle T has $\Gamma(T) \in R_S \times M_{S\times S}(R_S) = \left\{\begin{array}{c|c} \omega & S \\ \hline S & A\end{array}\right\}$ with $R_S := \mathbb{Z}(\{t_a\colon a \in S\})$:

$$\left({}_a\!\nwarrow_b, {}_b\!\nwarrow_a\right) \to \begin{array}{c|cc} 1 & a & b \\ \hline a & 1 & 1-t_a^{\pm 1} \\ b & 0 & t_a^{\pm 1}\end{array} \qquad T_1 \sqcup T_2 \to \begin{array}{c|cc} \omega_1\omega_2 & S_1 & S_2 \\ \hline S_1 & A_1 & 0 \\ S_2 & 0 & A_2\end{array}$$

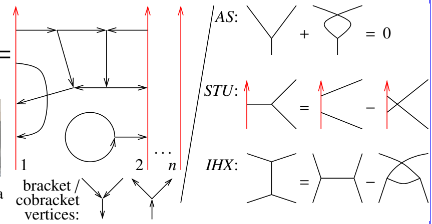
$$\begin{array}{c|ccc} \omega & a & b & S \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi \end{array} \xrightarrow[t_a, t_b \to t_c]{m_c^{ab}} \left(\begin{array}{c|cc} (1-\beta)\omega & c & S \\ \hline c & \gamma + \frac{\alpha\delta}{1-\beta} & \epsilon + \frac{\delta\theta}{1-\beta} \\ S & \phi + \frac{\alpha\psi}{1-\beta} & \Xi + \frac{\psi\theta}{1-\beta}\end{array}\right)$$

For long knots, ω is Alexander, and that's the fastest Alexander algorithm I know!

Dunfield: 1000-crossing fast.

Theorem [EK, Ha, En, Se]. There is a "homomorphic expansion"

$$Z\colon \left\{\begin{array}{c} S\text{-component} \\ (v/b\text{-})\text{tangles}\end{array}\right\} \to \mathcal{A}_S^v :=$$

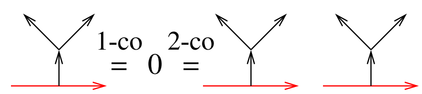


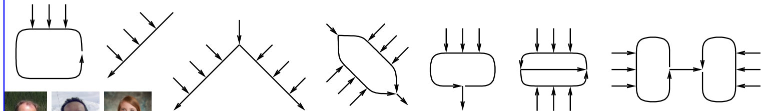

Etingof Kazhdan Haviv Enriquez Ševera

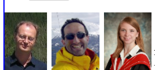
(it is enough to know Z on ⤡ and have disjoint union and stitching formulas) ...exponential and too hard!

Idea. Look for "ideal" quotients of $\mathcal{A}_S^v$ that have poly-sized descriptions; ...specifically, limit the co-brackets.

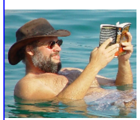
1-co and 2-co, aka TC and TC^2, on the right. The primitives that remain are:



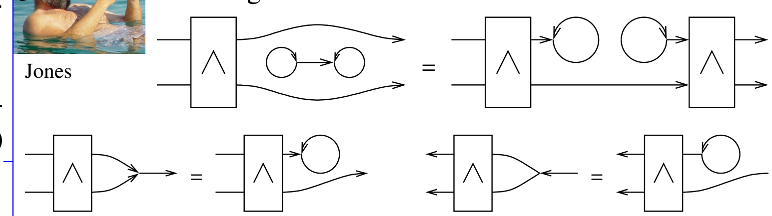



Alekseev, Torossian, Dancso

...manageable but still exponential!


Jones

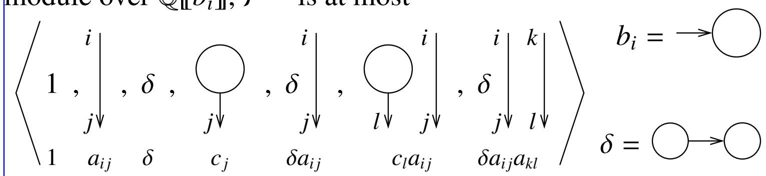
The 2D relations come from the relation with 2D Lie bialgebras:



We let $\mathcal{A}^{2,2}$ be $\mathcal{A}^v$ modulo 2-co and 2D, and $z^{2,2}$ be the projection of $\log Z$ to $\mathcal{P}^{2,2} := \pi\mathcal{P}^v$, where $\mathcal{P}^v$ are the primitives of $\mathcal{A}^v$.

Main Claim. $z^{2,2}$ is poly-time computable.

Main Point. $\mathcal{P}^{2,2}$ is poly-size, so how hard can it be? Indeed, as a module over $\mathbb{Q}[\![b_i]\!]$, $\mathcal{P}^{2,2}$ is at most



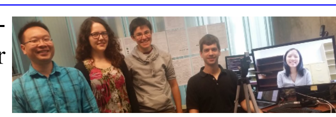
Claim. $R_{jk} = e^{a_{jk}}e^{\rho_{jk}}$ is a solution of the Yang-Baxter / R3 equation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ in $\exp\mathcal{P}^{2,2}$, with $\rho_{jk} :=$

$$\psi(b_j)\left(-c_k + \frac{c_k a_{jk}}{b_j} - \frac{\delta a_{jk}a_{jk}}{b_j^2}\right) + \frac{\phi(b_j)\psi(b_k)}{b_k\phi(b_k)}\left(c_k a_{kk} - \frac{\delta a_{jk}a_{kk}}{b_j}\right),$$

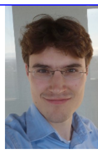
and with $\phi(x) := e^{-x} - 1 = -x + x^2/2 - \ldots$, and $\psi(x) := ((x+2)e^{-x} - 2 + x)/(2x) = x^2/12 - x^3/24 + \ldots$.

But how do we multiply in $\exp(\mathcal{P}^{2,2})$? How do we stitch?

Videos of a 4-hour version of this talk are at ωεβ/LD. Videos of private seminar meetings are at ωεβ/PP.



Many thanks: Vo, Halacheva, Dalvit, Ens, Lee (van der Veen, Schaveling)

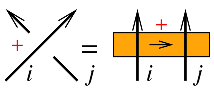

van der Veen

1-Smidgen sl_2 (with van der Veen). Let $\mathfrak{g}_1$ be the 4D Lie algebra $\mathfrak{g}_1 = \langle b,c,u,w\rangle$ over $\mathbb{Q}[\epsilon]/(\epsilon^2=0)$, with b central and $[w,c]=w,\ [c,u]=u$, and $[u,w]=b-2\epsilon c$, with CYBE $r_{ij}=(b_i-\epsilon c_i)c_j+u_iw_j$ in $\mathcal{U}(\mathfrak{g}_1)^{\otimes\{i,j\}}$. Over $\mathbb{Q}$, $\mathfrak{g}_1$ is a solvable approximation of sl_2: $\mathfrak{g}_1\supset\langle b,u,w,\epsilon b,\epsilon c,\epsilon u,\epsilon w\rangle\supset\langle b,\epsilon b,\epsilon c,\epsilon u,\epsilon w\rangle\supset 0$. In a certain sense, $\mathfrak{g}_1$ is more valuable than sl_2. (note: $\deg(b,c,u,w,\epsilon)=(1,0,1,0,1)$)

0-Smidgen sl_2 ☺. Let $\mathfrak{g}_0$ be $\mathfrak{g}_1$ at $\epsilon=0$, or $\mathbb{Q}\langle b,c,u,w\rangle/([b,\cdot]=0,\ [c,u]=u,\ [c,w]=-w,\ [u,w]=b$ with $r_{ij}=b_ic_j+u_iw_j$. It is $\mathfrak{a}^*\rtimes\mathfrak{a}$ where $\mathfrak{a}$ is the 2D Lie algebra $\mathbb{Q}\langle b,u\rangle$ and (c,w) is the dual basis of (b,u). It is even more valuable than $\mathfrak{g}_1$, but topology already got by other means almost everything $\mathfrak{g}_0$ has to give.

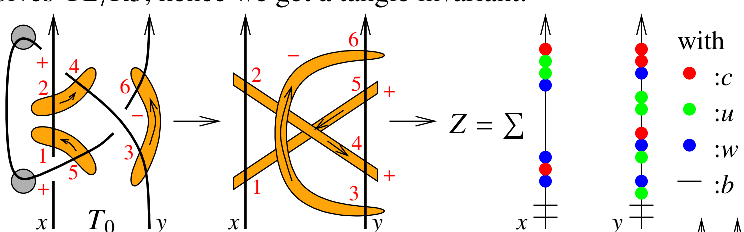
How did these arise? $sl_2=\mathfrak{b}^+\oplus\mathfrak{b}^-/\mathfrak{h}=:sl_2^+/\mathfrak{h}$, where $\mathfrak{b}^+=\langle c,w\rangle/[w,c]=w$ is a Lie bialgebra with $\delta\colon\mathfrak{b}^+\to\mathfrak{b}^+\otimes\mathfrak{b}^+$ by $\delta\colon(c,w)\mapsto(0,c\wedge w)$. Going back, $sl_2^+=\mathcal{D}(\mathfrak{b}^+)=(\mathfrak{b}^+)^*\oplus\mathfrak{b}^+=\langle b,u,c,w\rangle/\cdots$. **Idea.** Replace $\delta\to\epsilon\delta$ over $\mathbb{Q}[\epsilon]/(\epsilon^{k+1}=0)$. At $k=0$, get $\mathfrak{g}_0$. At $k=1$, get $[w,c]=w,\ [w,b']=-\epsilon w,\ [c,u]=u,\ [b',u]=-\epsilon u,\ [b',c]=0$, and $[u,w]=b'-\epsilon c$. Now note that $b'+\epsilon c$ is central, so switch to $b:=b'+\epsilon c$. This is $\mathfrak{g}_1$.

0-Smidgen Invariants. $r=Id\in\mathfrak{b}^-\otimes\mathfrak{b}^+$ solves the CYBE $[r_{12},r_{13}]+[r_{12},r_{23}]+[r_{13},r_{23}]=0$ in $\mathcal{U}(\mathfrak{g}_0)^{\otimes 3}$ and, by luck,

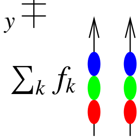


$=R_{ij}=e^{r_{ij}}=e^{b_ic_j+u_iw_j}\in\mathcal{U}(\mathfrak{g}_{0,i}\oplus\mathfrak{g}_{0,j})$

solves YB/R3, hence we get a tangle invariant:



Goal. Sort Z to be as on the right, with $f_k\in\mathbb{Q}[\![b_i]\!]$. Better, with $\zeta\in\mathbb{Q}[\![b_x,c_x,u_x,w_x,b_y,c_y,u_y,w_y]\!]$, write



$$Z=\mathbb{O}\left(\zeta|x\colon c_xu_xw_x,\ y\colon c_yu_yw_y\right)\qquad\text{(cuw form)}$$

Here $\mathbb{O}(poly\mid specs)$ plants the variables of *poly* in $\mathcal{S}(\oplus_i\mathfrak{g})$ on several tensor copies of $\mathcal{U}(\mathfrak{g})$ according to *specs*. E.g.,

$$\mathbb{O}\left(c_1^3u_1c_2e^{u_3}w_3^9|x\colon w_3c_1,\ y\colon u_1u_3c_2\right)=w^9c^3\otimes ue^uc\in\mathcal{U}(\mathfrak{g})_x\otimes\mathcal{U}(\mathfrak{g})_y$$

Lemma. $R_{ij}=e^{b_ic_j+u_iw_j}=\mathbb{O}\left(\exp\left(b_ic_j+\frac{e^{b_i}-1}{b_i}u_iw_j\right)|i\colon u_i,\ j\colon c_jw_j\right)$
$=\sum_{m,n}\frac{b_i^{m-n}(e^{b_i}-1)^n}{m!n!}u^n\otimes c^mw^n.$

Example. $Z(T_0)=$

$$\mathbb{O}\Bigg(1\exp\left(b_5c_1+\frac{e^{b_5}-1}{b_5}u_5w_1+b_2c_4+\frac{e^{b_2}-1}{b_2}u_2w_4-b_3c_6+\frac{e^{-b_3}-1}{b_3}u_3w_6\right)\Bigg|$$

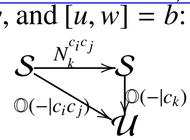
$x\colon c_1w_1u_2,\ y\colon u_3c_4w_4u_5c_6w_6\Big)$

$\mathbb{O}(\omega e^{L+Q})$: L bilinear in b_i and c_i, and Q a balanced quadratic in u_i and w_i with coefficients in $\mathbb{Q}(b_i,e^{b_i})\ni\omega$. "Admissible"

$=\mathbb{O}\left(?|x\colon c_xu_xw_x,\ y\colon c_yu_yw_y\right)$

The Big $\mathfrak{g}_0$ Lemma. Under $[c,u]=u$, $[c,w]=-w$, and $[u,w]=b$:

1. $N_k^{c_ic_j}:=\mathbb{O}(\zeta|c_ic_j)\overset{\to}{=}\mathbb{O}(\zeta/(c_i,c_j\to c_k)|c_k)$
(Meaning, $N_k^{c_ic_j}\colon\zeta\mapsto(\zeta/(c_i,c_j\to c_k))$ and the diagram commutes. Trivial, also for b, u, w.)

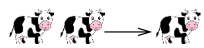


2a. $N^{uc}:=\mathbb{O}(e^{\gamma c+\beta u}|uc)\overset{\to}{=}\mathbb{O}(e^{\gamma c+e^{-\gamma}\beta u}|cu)$ (means $e^{\beta u}e^{\gamma c}=e^{\gamma c}e^{e^{-\gamma}\beta u}$
2b. $N^{wc}:=\mathbb{O}(e^{\gamma c+\alpha w}|wc)\overset{\to}{=}\mathbb{O}(e^{\gamma c+e^{\gamma}\alpha w}|cw)$ …in the $\{ax+b\}$ group)
3. $\mathbb{O}(e^{\alpha w+\beta u}|wu)=\mathbb{O}(e^{-b\alpha\beta+\alpha w+\beta u}|uw)$ (the Weyl relations)
4. $\mathbb{O}(e^{\delta uw}|wu)e^{\beta u}=e^{\nu\beta u}\mathbb{O}(e^{\delta uw}|wu)$, with $\nu=(1+b\delta)^{-1}$
(a. expand and crunch. b. use $w=b\hat{x}$, $u=\partial_x$. c. use "scatter and glow".)
5. $\mathbb{O}(e^{\delta uw}|wu)=\mathbb{O}(\nu e^{\nu\delta uw}|uw)$ (same techniques)
6. $N^{wu}:=\mathbb{O}(e^{\beta u+\alpha w+\delta uw}|wu)\overset{\to}{=}\mathbb{O}(\nu e^{-b\nu\alpha\beta+\nu\alpha w+\nu\beta u+\nu\delta uw}|uw)$

Sneaky. α may contain (other) u's, β may contain (other) w's.

Strand Stitching, m_k^{ij}, is defined as the composition

$$c_iu_i\,\overline{w_ic_j}\,u_jw_j\xrightarrow{N_k^{w_ic_j}}c_i\,\overline{u_ic_k}\,\overline{w_ku_j}\,w_j\xrightarrow{N_k^{u_ic_k}/\!/N_k^{w_ku_j}}\overline{c_ic_k}\,\overline{u_ku_k}\,\overline{w_kw_j}\xrightarrow{N_k^{c_ic_k}/\!/-\!-/\!/N_k^{w_kw_j}}c_ku_kw_k$$



1-Smidgen Invariants. Much is the same:
The Big $\mathfrak{g}_1$ Lemma. Parts 1 and 2 are the same, yet
6. $\mathbb{O}\left(e^{\alpha w+\beta u+\delta uw}|wu\right)=\mathbb{O}\left(\nu(1+\epsilon\nu\Lambda)e^{\nu(-b\alpha\beta+\alpha w+\beta u+\delta uw)}|cuw\right)$

Here Λ is for Λόγος, "a principle of order and knowledge", a balanced quartic in α, β, c, u, and w:

$$\begin{aligned}\Lambda=&-b\nu\left(\nu^2\alpha^2\beta^2+4\delta\nu\alpha\beta+2\delta^2\right)/2-\delta\nu^3(3b\delta+2)\beta^2u^2/2\\&-b\delta^4\nu^3u^2w^2/2-\delta^2\nu^3(2b\delta+1)\beta u^2w\\&-\nu^2(2b\delta+1)(\nu\alpha\beta+2\delta)\beta u-2b\delta^2\nu^2(\nu\alpha\beta+\delta)uw\\&+\delta\nu^3(b\delta+2)\alpha^2w^2/2+2(\nu\alpha\beta+\delta)c+2\delta\nu\beta cu+2\delta^2\nu cuw\\&+2\delta\nu\alpha cw+\delta^2\nu^3\alpha uw^2+\nu^2(\nu\alpha\beta+2\delta)\alpha w.\end{aligned}$$

Proof. A brutal hell.
Problem. We now need to normal-order perturbed Gaussians!
Solution. Borrow some tactics from QFT:

$$\mathbb{O}(\epsilon P(c,u)e^{\gamma c+\beta u}|uc)=\mathbb{O}(\epsilon P(\partial_\gamma,\partial_\beta)e^{\gamma c+\beta u}|uc)=$$
$$\mathbb{O}(\epsilon P(\partial_\gamma,\partial_\beta)e^{\gamma c+e^{-\gamma}\beta u}|cu),$$

and likewise

$$\mathbb{O}\left(\epsilon P(u,w)e^{\alpha w+\beta u+\delta uw}|wu\right)=\mathbb{O}\left(\epsilon P(\partial_\beta,\partial_\alpha)\nu e^{\nu(-b\alpha\beta+\alpha w+\beta u+\delta uw)}|cuw\right)$$

Note. Strand stitching requires a tiny extra step.
Finally, the values of the generators ⤯, ⤰, $\overrightarrow{n}$, $\overleftarrow{n}$, $\underset{\to}{u}$, and $\underset{\leftarrow}{u}$, are set by brutally solving many equations, non-uniquely.

Pragmatic Simplifications. Get rid of $\zeta=(e^b-1)/b$ factors by rescaling $u\to\bar{u}=\zeta u$. Complement this with $\beta\to\bar{\beta}=\zeta^{-1}\beta$, $\delta\to\bar{\delta}=\zeta^{-1}\delta$, $\epsilon\to\bar{\epsilon}=\zeta^{-1}\epsilon$. Simplify further by naming $e^b\to t$; e.g., $\nu\to\bar{\nu}=(1+(t-1)\delta)^{-1}$. Get confused by renaming $(\bar{u},\bar{\beta},\bar{\delta},\bar{\nu})\to(u,\beta,\delta,\nu)$, and more confused by working with $\mu=\nu^{-1}$ and $\mathbb{E}(\omega,L,Q,P):=\omega^{-1}(1+\epsilon\omega^{-4}P)e^{L+\omega^{-1}Q}$, where $\omega\in R:=\mathbb{Q}(t_k)$, $L=\sum l_{ij}b_ic_j$ with $l_{ij}\in\mathbb{Z}$, $Q=\sum q_{ij}u_iw_j$ with $q_{ij}\in R$, and P is a balanced quartic polynomial in c_i, u_i, and w_i with coefficients in R. Magically, all coefficients are now Laurent polynomials in the t_k's.

Rough complexity estimate, after $t_k\to t$. n: xing number; w: width, maybe $\sim\sqrt{n}$.

$$\underbrace{n}_{A}\overbrace{\sum_{d=0}^{4}\underbrace{w^{4-d}}_{E}\underbrace{w^d}_{F}\underbrace{n^2}_{G}}^{B}=n^3w^4\in[n^5,n^7]$$

A: go over stitchings in order. B: multiplication ops per $N^{u_iw_j}$. d: deg of u_i,w_j in P. E: #terms of deg d in P. F: ops per term. G: cost per polynomial multiplication op.

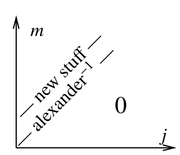
Expectation. Our invariant is the "1-higher diagonal" in the MMR expansion of the coloured Jones polynomial J_λ.


Melvin, Morton, Garoufalidis

Theorem ([BNG], conjectured [MM], elucidated [Ro]). Let $J_d(K)$ be the coloured Jones polynomial of K, in the d-dimensional representation of $sl(2)$. Writing

$$\left.\frac{(q^{1/2}-q^{-1/2})J_d(K)}{q^{d/2}-q^{-d/2}}\right|_{q=e^\hbar}=\sum_{j,m\geq 0}a_{jm}(K)d^j\hbar^m,$$

"below diagonal" coefficients vanish, $a_{jm}(K)=0$ if $j>m$, and "on diagonal" coefficients give the inverse of the Alexander polynomial: $\left(\sum_{m=0}^{\infty}a_{mm}(K)\hbar^m\right)\cdot A(K)(e^\hbar)=1$.



Demo Programs for 0-Co. ωεβ/Demo

The R-matrices

$R^+_{0,i\_,j\_} := \mathbb{E}\left[b_i c_j + b_i^{-1}\left(e^{b_i}-1\right) u_i w_j\right];$

$R^-_{0,i\_,j\_} := \mathbb{E}\left[-b_i c_j + b_i^{-1}\left(e^{-b_i}-1\right) u_i w_j\right];$

Utilities

$\text{CF}[\omega\_ .\ \mathbb{E}[Q\_]] := \text{Simplify}[\omega]\ \mathbb{E}[\text{Simplify}[Q]];$

$\mathbb{E}\ /{:}\ \mathbb{E}[Q1\_]\ \mathbb{E}[Q2\_] := \text{CF@}\mathbb{E}[Q1 + Q2];$

$\omega1\_ .\ \mathbb{E}[Q1\_] \equiv \omega2\_ .\ \mathbb{E}[Q2\_] := \text{Simplify}[\omega1 == \omega2 \wedge Q1 == Q2];$

Normal Ordering Operators

$N_{u_{i\_}\,c_{j\_}\to k\_}[\omega\_ .\ \mathbb{E}[Q\_]] := \text{CF}\big[$
$\quad \omega\,\mathbb{E}[e^{-\gamma}\beta\,u_k + \gamma\,c_k + (Q\ /.\ c_j \mid u_i \to 0)]\ /.\ \{\gamma \to \partial_{c_j}Q,\ \beta \to \partial_{u_i}Q\}\big];$

$N_{w_{i\_}\,c_{j\_}\to k\_}[\omega\_ .\ \mathbb{E}[Q\_]] := \text{CF}\big[$
$\quad \omega\,\mathbb{E}[e^{\gamma}\alpha\,w_k + \gamma\,c_k + (Q\ /.\ c_j \mid w_i \to 0)]\ /.\ \{\gamma \to \partial_{c_j}Q,\ \alpha \to \partial_{w_i}Q\}\big];$

$N_{w_{i\_}\,u_{j\_}\to k\_}[\omega\_ .\ \mathbb{E}[Q\_]] := \text{CF}\big[$
$\quad \nu\,\omega\,\mathbb{E}[-b_k\,\nu\,\alpha\,\beta + \nu\,\beta\,u_k + \nu\,\delta\,u_k\,w_k + \nu\,\alpha\,w_k + (Q\ /.\ w_i \mid u_j \to 0)]\ /.$
$\quad \nu \to (1 + b_k\,\delta)^{-1}\ /.$
$\quad \{\alpha \to \partial_{w_i}Q\ /.\ u_j \to 0,\ \beta \to \partial_{u_j}Q\ /.\ w_i \to 0,\ \delta \to \partial_{w_i,u_j}Q\}\big];$

Stitching

$m_{i\_,j\_\to k\_}[\omega\_ .\ \mathbb{E}[Q\_]] := \text{CF}\big[\text{Module}\big[\{x\},$
$\quad \left(\omega\,\mathbb{E}[Q]\ /.\ b_{i|j} \to b_k\ //\ N_{w_i\,c_j\to x}\ //\ N_{u_i\,c_x\to x}\ //\ N_{w_x\,u_j\to x}\right)\ /.$
$\quad \{c_i \to c_k,\ w_j \to w_k,\ y\_{x} :\!\to y_k\}\big]\big]$

Some calculations for T_0

$T_{0,0} = R^+_{0,5,1}\,R^+_{0,2,4}\,R^-_{0,3,6}$

$\mathbb{E}\left[b_5 c_1 + b_2 c_4 - b_3 c_6 + \frac{(-1+e^{b_5})\,u_5 w_1}{b_5} + \frac{(-1+e^{b_2})\,u_2 w_4}{b_2} + \frac{(-1+e^{-b_3})\,u_3 w_6}{b_3}\right]$

$T_{0,1} = T_{0,0}\ //\ N_{u_3\,c_4\to 4}$

$\mathbb{E}\left[b_5 c_1 + b_2 c_4 - b_3 c_6 + \frac{(-1+e^{b_5})\,u_5 w_1}{b_5} + \frac{(-1+e^{b_2})\,u_2 w_4}{b_2} + \frac{e^{-b_2}(-1+e^{-b_3})\,u_4 w_6}{b_3}\right]$

$T_{0,2} = T_{0,1}\ //\ N_{w_4\,u_5\to 4}$

$\mathbb{E}\left[b_5 c_1 + b_2 c_4 + \frac{(-1+e^{b_5})\left(-(-1+e^{b_2})\,b_4 u_2 + b_2 u_4\right) w_1}{b_2 b_5} + \frac{(-1+e^{b_2})\,u_2 w_4}{b_2} - \frac{b_3^2 c_6 + e^{-b_2-b_3}(-1+e^{b_3})\,u_4 w_6}{b_3}\right]$

$T_{0,2}\ //\ N_{w_1\,u_2\to 1}$

$\frac{1}{1-\frac{(-1+e^{b_2})(-1+e^{b_5})\,b_1 b_4}{b_2 b_5}}\ \mathbb{E}\Big[\frac{1}{b_3\left((-1+e^{b_2})(-1+e^{b_5})\,b_1 b_4 - b_2 b_5\right)}$
$\quad \big(b_3 b_5\left((-1+e^{b_2})(-1+e^{b_5})\,b_1 b_4 - b_2 b_5\right) c_1 +$
$\quad b_2 b_3\left((-1+e^{b_2})(-1+e^{b_5})\,b_1 b_4 - b_2 b_5\right) c_4 +$
$\quad (-1+e^{b_2})(-1+e^{b_5})\,b_3 b_4 u_1 w_1 - (-1+e^{b_5})\,b_2 b_3 u_4 w_1 -$
$\quad (-1+e^{b_2})\,b_3 b_5 u_1 w_4 + (-1+e^{b_2})(-1+e^{b_5})\,b_1 b_3 u_4 w_4 -$
$\quad \left((-1+e^{b_2})(-1+e^{b_5})\,b_1 b_4 - b_2 b_5\right)$
$\quad \left(b_3^2 c_6 + e^{-b_2-b_3}(-1+e^{b_3})\,u_4 w_6\right)\big)\Big]$

$T_{0,0}\ //\ m_{1,2\to 1}\ //\ m_{3,4\to 3}\ //\ m_{3,5\to 3}\ //\ m_{3,6\to 3}$

$\frac{1}{1-(-1+e^{b_1})(-1+e^{b_3})}\mathbb{E}\Big[b_3 c_1 + b_1 c_3 - b_3 c_3 +$
$\quad \frac{(-1+e^{b_1})(-1+e^{b_3})\,u_1 w_1}{(-e^{b_1}-e^{b_3}+e^{b_1+b_3})\,b_1} - \frac{e^{-b_3}(-1+e^{b_1})\left(b_3 u_1 - e^{b_3}(-1+e^{b_3})\,b_1 u_3\right) w_3}{(-e^{b_1}-e^{b_3}+e^{b_1+b_3})\,b_1 b_3} +$
$\quad \frac{e^{-b_1}(-1+e^{b_3})\,u_3\left(-e^{b_1+b_3} w_1 + (e^{b_1}+e^{b_3}-e^{b_1+b_3})\,w_3\right)}{(-e^{b_1}-e^{b_3}+e^{b_1+b_3})\,b_3}\Big]$

Verifying meta-associativity

$Q0 = \mathbb{E}[\text{Sum}[f_i\,c_i,\ \{i, 3\}] + \text{Sum}[f_{i,j}\,u_i\,w_j,\ \{i, 3\},\ \{j, 3\}]]$

$\mathbb{E}[c_1 f_1 + c_2 f_2 + c_3 f_3 + u_1 w_1 f_{1,1} + u_1 w_2 f_{1,2} + u_1 w_3 f_{1,3} + u_2 w_1 f_{2,1} +$
$\quad u_2 w_2 f_{2,2} + u_2 w_3 f_{2,3} + u_3 w_1 f_{3,1} + u_3 w_2 f_{3,2} + u_3 w_3 f_{3,3}]$

$(Q0\ //\ m_{1,2\to 1}\ //\ m_{1,3\to 1}) \equiv (Q0\ //\ m_{2,3\to 2}\ //\ m_{1,2\to 1})$

True

Testing R3

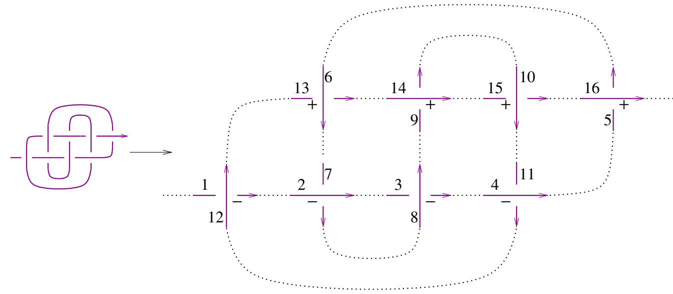
$t1 = R^+_{0,1,2}\,R^+_{0,3,4}\,R^+_{0,5,6}\ //\ m_{3,5\to x}\ //\ m_{1,6\to y}\ //\ m_{2,4\to z}$

$\mathbb{E}\left[b_x\,(c_y + c_z) + \frac{(-1+e^{b_x})\,u_x\,(w_y + w_z)}{b_x} + \frac{b_y^2 c_z + (-1+e^{b_y})\,u_y w_z}{b_y}\right]$

$t1 \equiv (R^+_{0,1,2}\,R^+_{0,3,4}\,R^+_{0,5,6}\ //\ m_{1,3\to x}\ //\ m_{2,5\to y}\ //\ m_{4,6\to z})$

True

8_{17}



$z1 = R^-_{0,12,1}\,R^-_{0,2,7}\,R^-_{0,8,3}\,R^-_{0,4,11}\,R^+_{0,16,5}\,R^+_{0,6,13}\,R^+_{0,14,9}\,R^+_{0,10,15};$

$\text{Do}[z1 = (z1\ //\ m_{1,n\to 1})\ /.\ b\_ \to b,\ \{n, 2, 16\}];$

$\{\text{CF@}z1,\ \text{KnotData}[\{8, 17\},\ \text{"AlexanderPolynomial"}][t]\}$

$\left\{-\frac{e^{3b}\,\mathbb{E}[0]}{1-4e^{b}+8e^{2b}-11e^{3b}+8e^{4b}-4e^{5b}+e^{6b}},\ 11 - \frac{1}{t^3} + \frac{4}{t^2} - \frac{8}{t} - 8t + 4t^2 - t^3\right\}$

Demo Programs for 1-Co. ωεβ/Demo

The Λόγος

$\Lambda[k\_] := (1 - t_k)\,\left(\alpha^2\beta^2 + 4\,\alpha\,\beta\,\delta\,\mu + 2\,\delta^2\mu^2\right)/2 + 2\,\mu^2\,(\alpha\,\beta + \delta\,\mu)\,c_k -$
$\quad \beta\,(2\,\mu - 1)\,(\alpha\,\beta + 2\,\delta\,\mu)\,u_k + 2\,\beta\,\delta\,\mu^2\,c_k\,u_k - \beta^2\,\delta\,(3\,\mu - 1)\,u_k^2/2 +$
$\quad \alpha\,(\alpha\,\beta + 2\,\delta\,\mu)\,w_k + 2\,\alpha\,\delta\,\mu^2\,c_k\,w_k - 2\,(t_k - 1)\,\delta^2\,(\alpha\,\beta + \delta\,\mu)\,u_k\,w_k +$
$\quad 2\,\delta^2\mu^2\,c_k\,u_k\,w_k - \beta\,\delta^2\,(2\,\mu - 1)\,u_k^2\,w_k + \alpha^2\,\delta\,(1 + \mu)\,w_k^2/2 +$
$\quad \alpha\,\delta^2\,u_k\,w_k^2 - (t_k - 1)\,\delta^4\,u_k^2\,w_k^2/2;$

Differential Polynomials

$DP_{x\_\to D_{\alpha\_},\,y\_\to D_{\beta\_}}[P\_][f\_] :=$ (* means $P[\partial_\alpha, \partial_\beta][f]$ *)
$\quad \text{Total}[\text{CoefficientRules}[P,\ \{x, y\}]\ /.$
$\quad (\{m\_, n\_\} \to c\_) :\!\to c\,D[f,\ \{\alpha, m\},\ \{\beta, n\}]]$

Utilities

$\text{CF}[\mathbb{E}[\omega\_, L\_, Q\_, P\_]] := \text{Expand /@ Together /@}$
$\quad \mathbb{E}[\omega\ /.\ b_{\iota\_} :\!\to \text{Log}[t_\iota],\ L,\ Q\ /.\ b_{\iota\_} :\!\to \text{Log}[t_\iota],$
$\quad P\ /.\ b_{\iota\_} :\!\to \text{Log}[t_\iota]];$

$\mathbb{E}\ /{:}\ \mathbb{E}[\omega1\_, L1\_, Q1\_, P1\_]\ \mathbb{E}[\omega2\_, L2\_, Q2\_, P2\_] :=$
$\quad \text{CF@}\mathbb{E}\left[\omega1\,\omega2,\ L1 + L2,\ \omega2\,Q1 + \omega1\,Q2,\ \omega2^4\,P1 + \omega1^4\,P2\right];$

Normal Ordering Operators

$N_{u_{i\_}\,c_{j\_}\to k\_}[\mathbb{E}[\omega\_, L\_, Q\_, P\_]] := \text{With}\big[\{q = e^{-\gamma}\beta\,u_k + \gamma\,c_k\},\ \text{CF}\big[$
$\quad \mathbb{E}\big[\omega,\ \gamma\,c_k + (L\ /.\ c_j \to 0),\ \omega\,e^{-\gamma}\beta\,u_k + (Q\ /.\ u_i \to 0),$
$\quad e^{-q}\,DP_{c_j\to D_\gamma,\,u_i\to D_\beta}[P][e^q]\big]\ /.\ \{\gamma \to \partial_{c_j}L,\ \beta \to \omega^{-1}\,\partial_{u_i}Q\}\big]\big];$

$N_{w_{i\_}\,c_{j\_}\to k\_}[\mathbb{E}[\omega\_, L\_, Q\_, P\_]] := \text{With}\big[\{q = e^{\gamma}\alpha\,w_k + \gamma\,c_k\},\ \text{CF}\big[$
$\quad \mathbb{E}\big[\omega,\ \gamma\,c_k + (L\ /.\ c_j \to 0),\ \omega\,e^{\gamma}\alpha\,w_k + (Q\ /.\ w_i \to 0),$
$\quad e^{-q}\,DP_{c_j\to D_\gamma,\,w_i\to D_\alpha}[P][e^q]\big]\ /.\ \{\gamma \to \partial_{c_j}L,\ \alpha \to \omega^{-1}\,\partial_{w_i}Q\}\big]\big];$

$N_{w_{i\_}\,u_{j\_}\to k\_}[\mathbb{E}[\omega\_, L\_, Q\_, P\_]] :=$
$\quad \text{With}\big[\{q = (1 - t_k)\,\mu^{-1}\alpha\,\beta + \mu^{-1}\beta\,u_k + \mu^{-1}\delta\,u_k\,w_k + \mu^{-1}\alpha\,w_k\},\ \text{CF}\big[$
$\quad \mathbb{E}\big[\mu\,\omega,\ L,\ \mu\,\omega\,q + \mu\,(Q\ /.\ w_i \mid u_j \to 0),$
$\quad \mu^4\,e^{-q}\,DP_{w_i\to D_\alpha,\,u_j\to D_\beta}[P][e^q] + \omega^4\,\Lambda[k]\big]\ /.$
$\quad \mu \to 1 + (t_k - 1)\,\delta\ /.$
$\quad \{\alpha \to \omega^{-1}\,(\partial_{w_i}Q\ /.\ u_j \to 0),\ \beta \to \omega^{-1}\,(\partial_{u_j}Q\ /.\ w_i \to 0),$
$\quad \delta \to \omega^{-1}\,\partial_{w_i,u_j}Q\}\big]\big];$

Stitching

$m_{i\_,j\_\to k\_}[Z\_] := \text{Module}\big[\{x, y, z\},$
$\quad Z\ //\ N_{w_i\,c_j\to x}\ //\ N_{w_x\,u_j\to y}\ //\ \text{ReplaceAll}[\{c_{x|y} \to c_x,\ w_j \to w_y\}]\ //$
$\quad N_{u_i\,c_x\to x}\ //\ \text{ReplaceAll}\left[z\_{i|j|x|y} \to z_k\right]\ //\ \text{CF}\big]$

The Generators

$$R^+_{i\_,j\_} := \mathbb{E}\Big[1,\ b_i c_j,\ u_i w_j,\ -c_i (t_i-1)^2/2 - c_i^2 (t_i-1)^2/2 + c_i c_j \left(t_j^2 - t_i - 2\right)/2 - c_j u_i w_i/2 + c_i (1-t_i) u_i w_i - u_i^2 w_i^2/2 + u_i w_j + c_j t_i u_i w_j/2 + c_i (t_i-2) t_i u_i w_j + c_i (1+t_j) u_j w_j/2 + (t_i-1) u_i^2 w_i w_j - (t_i-2) t_i u_i^2 w_j^2/2\Big];$$

$$R^-_{i\_,j\_} := \mathbb{E}\Big[1,\ -b_i c_j,\ -t_i^{-1} u_i w_j,\ c_i (t_i-1)^2/2 + c_i^2 (t_i-1)^2/2 + c_i c_j \left(2 + t_i - t_j^2\right)/2 + c_j u_i w_i/2 + c_i (t_i-1) u_i w_i + u_i^2 w_i^2/2 + \left(1 - t_i^{-1}\right) u_i w_j/2 + c_i \left(2 t_i - 5 + 3 t_i^{-1}\right) u_i w_j/2 + c_j \left(t_i^{-1} + 1 - t_i^{-1} t_j^2\right) u_i w_j/2 - c_i (t_j+1) u_j w_j/2 + \left(2 - 3 t_i^{-1}\right) u_i^2 w_i w_j/2 + \left(1 + 2 t_i^{-2} - 3 t_i^{-1}\right) u_i^2 w_j^2/2 - t_i^{-1} (1+t_j) u_i u_j w_j^2/2\Big];$$

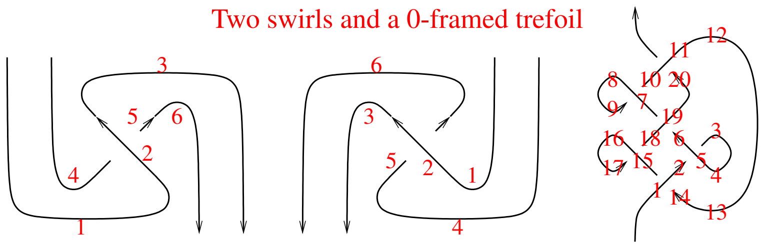
$$ur_{i\_} := \mathbb{E}\left[t_i^{-1/4}, 0, 0, c_i t_i/4 + u_i w_i/8\right];$$

$$nr_{i\_} := \mathbb{E}\left[t_i^{1/4}, 0, 0, -c_i t_i^3/4 - t_i^2 u_i w_i/8\right];$$

$$ul_{i\_} := \mathbb{E}\left[t_i^{1/4}, 0, 0, c_i t_i (4+t_i)/4 - t_i^2 u_i w_i/8\right];$$

$$nl_{i\_} := \mathbb{E}\left[t_i^{-1/4}, 0, 0, -c_i \left(1 + 4 t_i^{-1}\right)/4 + u_i w_i/8\right];$$

Two swirls and a 0-framed trefoil



$$t2 = ur_1 R^-_{2,5} nr_3 ur_4 nr_6 \mathbin{//} m_{1,2\to1} \mathbin{//} m_{1,3\to1} \mathbin{//} m_{4,5\to4} \mathbin{//} m_{4,6\to4}$$

$$\mathbb{E}\Big[1,\ -b_1 c_4,\ -\frac{u_1 w_4}{t_1},\ \frac{c_1}{2} + \frac{c_1^2}{2} + c_1 c_4 - c_1 t_1 - c_1^2 t_1 + \frac{1}{2} c_1 c_4 t_1 + \frac{1}{2} c_1 t_1^2 + \frac{1}{2} c_1^2 t_1^2 - \frac{1}{2} c_1 c_4 t_4^2 - c_1 u_1 w_1 + \frac{1}{2} c_4 u_1 w_1 + c_1 t_1 u_1 w_1 + \frac{1}{2} u_1^2 w_1^2 + \frac{3 u_1 w_4}{8} - \frac{5}{2} c_1 u_1 w_4 + \frac{1}{2} c_4 u_1 w_4 - \frac{u_1 w_4}{2 t_1} + \frac{3 c_1 u_1 w_4}{2 t_1} + \frac{c_4 u_1 w_4}{2 t_1} - \frac{1}{8} t_1 u_1 w_4 + c_1 t_1 u_1 w_4 + \frac{t_4 u_1 w_4}{8 t_1} + \frac{t_4^2 u_1 w_4}{8 t_1} - \frac{c_4 t_4^2 u_1 w_4}{2 t_1} - \frac{1}{2} c_1 u_4 w_4 - \frac{1}{2} c_1 t_4 u_4 w_4 + u_1^2 w_1 w_4 - \frac{3 u_1^2 w_1 w_4}{2 t_1} + \frac{1}{2} u_1^2 w_4^2 + \frac{u_1^2 w_4^2}{t_1^2} - \frac{3 u_1^2 w_4^2}{2 t_1} - \frac{u_1 u_4 w_4^2}{2 t_1} - \frac{t_4 u_1 u_4 w_4^2}{2 t_1}\Big]$$

$$t2 == \left(ul_1 R^-_{2,5} nl_3 ul_4 nl_6 \mathbin{//} m_{1,2\to1} \mathbin{//} m_{1,3\to1} \mathbin{//} m_{4,5\to4} \mathbin{//} m_{4,6\to4}\right)$$

True

$$z2 = R^+_{1,14} R^-_{5,2} nr_3 ul_4 R^+_{19,6} R^-_{7,10} nl_8 ur_9 R^+_{11,20} nr_{12} ul_{13} R^-_{15,18} nl_{16} ur_{17};$$

$$(\text{Do}[z2 = z2 \mathbin{//} m_{1,k\to1},\ \{k, 2, 20\}];\ z2 = z2\ /.\ a\_{}_1 :\!\to a)$$

$$\mathbb{E}\Big[-1 + \frac{1}{t} + t,\ 0,\ 0,\ -16 + \frac{9c}{2} - \frac{2c}{t^4} + \frac{1}{t^3} + \frac{11c}{2t^3} - \frac{4}{t^2} - \frac{8c}{t^2} + \frac{10}{t} + \frac{4c}{t} + 18t - 10ct - 14t^2 + 8ct^2 + 7t^3 - \frac{3ct^3}{2} - 2t^4 - 2ct^4 + 2ct^5 - \frac{ct^6}{2} - 4uw + \frac{2uw}{t^4} - \frac{7uw}{2t^3} + \frac{9uw}{2t^2} + \frac{uw}{2t} + 6tuw - 2t^2uw - \frac{1}{2}t^3uw + \frac{3}{2}t^4uw - \frac{1}{2}t^5uw\Big]$$

```
FromCoefficientRules[
 CoefficientRules[z2[[4]], {c, u, w}] /.
  {(e_ → a_) :→ (e → Simplify[a])}, {c, u, w}]
```

$$-\frac{\left(1-t+t^2\right)^2 \left(-1+2t-3t^2+2t^3\right)}{t^3} - \frac{c\left(1-t+t^2\right)^3 \left(4+t-5t^2-t^3+t^4\right)}{2t^4} - \frac{\left(1-t+t^2\right)^3 \left(-4-5t+t^3\right) uw}{2t^4}$$

"God created the knots; all else in topology is the work of mortals."

Leopold Kronecker (modified)

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Questions and To Do List. • Clean up and write up. • Implement well, compute for everything in sight. • Why are our quantities polynomials rather than just rational functions? • Bounds on their degrees? • Find the 2-variable version (for knots). How complex is it? • What about links / closed components? • Fully digest the "expansion" theorem; include cuaps. • Explore the (non-)dependence on R. • Is there a canonical R? • What does "group like" mean? • Strand removal? Strand doubling? Strand reversal? • Say something about knot genus. • Find the EK/AT/KV "vertex". • Use as a playground to study associators/braidors. • Restate in topological language. • Study the associated (v-)braid representations. • Study mirror images and the $\mathfrak{b}^+ \leftrightarrow \mathfrak{b}^-$ involution. • Study ribbon knots. • Make precise the relationship with Γ-calculus and Alexander. • Relate to the coloured Jones polynomial. • Relate with "ordinary" q-algebra. • k-smidgen sl_n, etc. • Are there "solvable" CYBE algebras not arising from semi-simple algebras? • Categorify and appease the Gods.

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Disclaimer. This is all quite new. The overall picture is correct, yet some details might be somewhat off. Many pieces are certainly not in their final form yet. **Help Needed!**