

Knots in Three and Four Dimensions

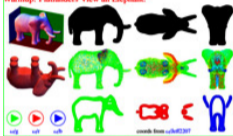
Dror Bar-Natan, <http://drorbn.net/syr21> (see “handout”!)

Syracuse by Web, April 2021

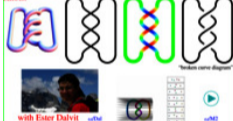
Abstract. Much as we can understand 3-dimensional objects by staring at their pictures and x-ray images and slices in 2-dimensions, so can we understand 4-dimensional objects by staring at their pictures and x-ray images and slices in 3-dimensions, capitalizing on the fact that we understand 3-dimensions pretty well. So we will spend some time staring at and understanding various 2-dimensional views of a 3-dimensional elephant, and then even more simply, various 2-dimensional views of some 3-dimensional knots. This achieved, we'll take the leap and visualize some 4-dimensional knots by their various traces in 3-dimensional space, and if we'll still have time, we'll prove that these knots are really knotted.

Abstract. Much as we can understand 3-dimensional objects by staring at their pictures and x-ray images and slices in 2-dimensions, so can we understand 4-dimensional objects by staring at their pictures and x-ray images and slices in 3-dimensions, capitalizing on the fact that we understand 3-dimensions pretty well. So we will spend some time staring at and understanding various 2-dimensional views of a 3-dimensional elephant, and then even more simply, various 2-dimensional views of some 3-dimensional knots. This achieved, we'll take the leap and visualize some 4-dimensional knots by their various traces in 3-dimensional space, and if we'll still have time, we'll prove that these knots are really knotted.

Warmup: Flatlanders View an Elephant.



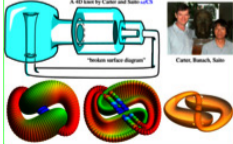
Knots.



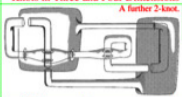
Formally, "a differentiable embedding of S^1 in R^3 modulo differentiable deformations of such".

2-Knots / 4D Knots.

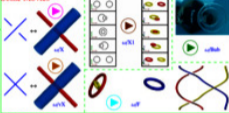
Formally, "a differentiable embedding of S^2 in R^4 modulo differentiable deformations of such".



Knots in Three and Four Dimensions



Some Movies



Some Unknots



Reidemeister's Theorem. (a) Every knot has a "broken curve diagram", made only of curves and "crossings" like \times . (b) Two knot diagrams represent the same 3D knot if they differ by a sequence of "Reidemeister moves".



3-Colourings. Colour the arcs of a broken arc diagram in RGB so that every crossing is either mono-chromatic or tri-chromatic. Let $A(K)$ be the number of such 3-colourings that K has.

Example. $A(\bigcirc) = 3$ while $A(\bigcirc) = 9$; so $\bigcirc \neq \bigcirc$.

Riddle. Is $A(K)$ always a power of 3?

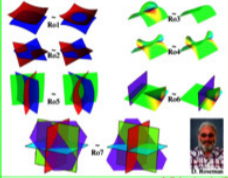
Proof sketch. It is enough to show that for each Reidemeister move, there is an end-colours-preserving bijection between the colourings of the two sides. E.g.:



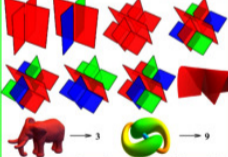
Theorem. Every 2-knot can be represented by a "broken surface diagram" made of the following basic ingredients,



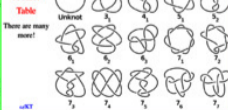
...and any two representations of the same knot differ by a sequence of the following "Roseman moves":



3-Colourings in 4D



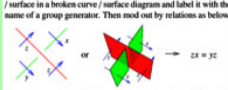
A Knot Table



There are many more!

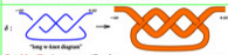
Knots in Three and Four Dimensions, 2

A Stronger Invariant. There is an assignment of groups to knots / 2-knots as follows. Put an arrow "under" every un-broken curve / surface in a broken curve / surface diagram and label it with the name of a group generator. Then mod out by relations as below.

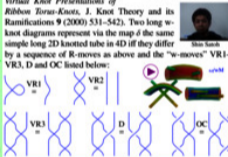


Facts. The resulting "Fundamental group" $\pi_1(K)$ of a knot / 2-knot K is a very strong but not very computable invariant of K . Though it has computable projections; e.g., for any finite G , count the homomorphisms from $\pi_1(K)$ to G .

Exercise. Show that $|\text{Hom}(\pi_1(K) \rightarrow S_3)| = A(K) + 3$.



Satoh's Conjecture. (Satoh, Virtual Knot Presentations of Ribbon Torus-Knots, J. Knot Theory and its Ramifications 9 (2000) 531-542). Two long knot diagrams represent via the map δ the same simple long 2D knotted tube in 4D if they differ by a sequence of R-moves as above and the "w-moves" VR1-VR3, D and OC listed below:



Some knot theory books.

- Colin C. Adams, *The Knot Book, an Elementary Introduction to the Mathematical Theory of Knots*, American Mathematical Society, 2004.
- Meike Akveld and Andrew Jobbings, *Knots Unravelled, from Strings to Mathematics*, Arbelos 2011.
- J. Scott Carter and Masahiko Saito, *Knotted Surfaces and Their Diagrams*, American Mathematical Society, 1997.
- Peter Cromwell, *Knots and Links*, Cambridge University Press, 2004.
- W.B. Raymond Lickorish, *An Introduction to Knot Theory*, Springer 1997.

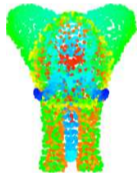
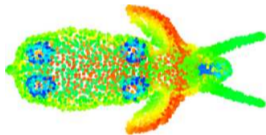
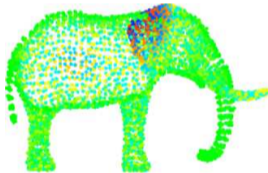
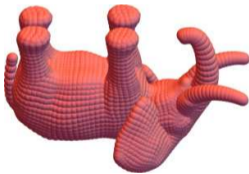
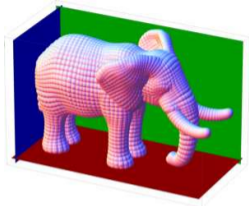


Thanks for inviting me to Syracuse! As most of you have never seen it, here's a picture of the lecture room:



If you can, please turn your video on! (And mic, whenever needed).

Warmup: Flatlanders View an Elephant.



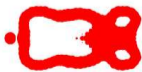
ω/g



ω/r

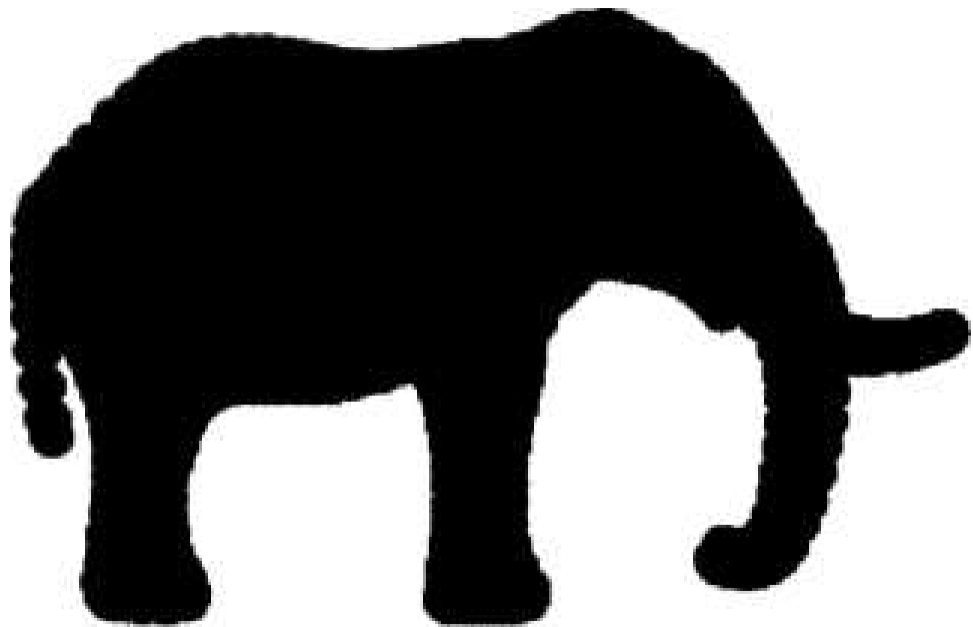


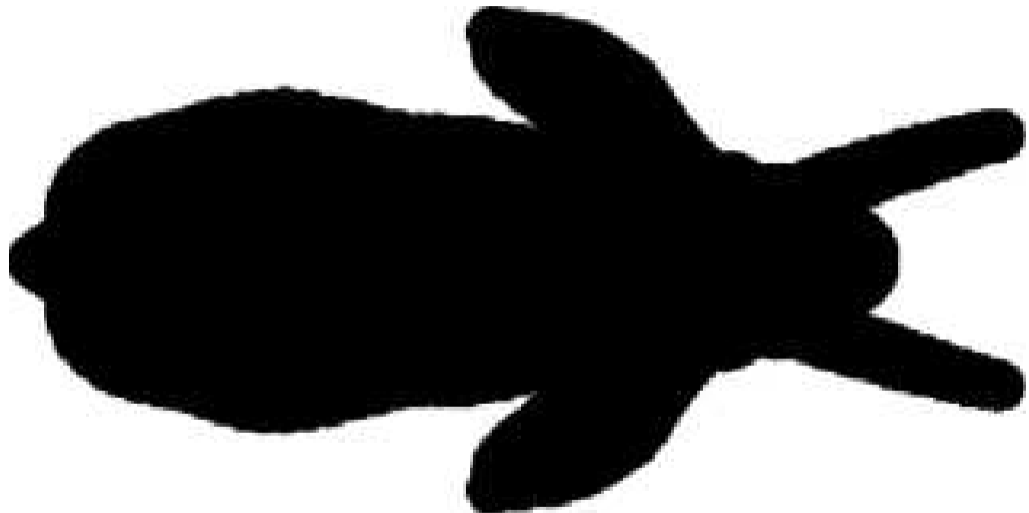
ω/b



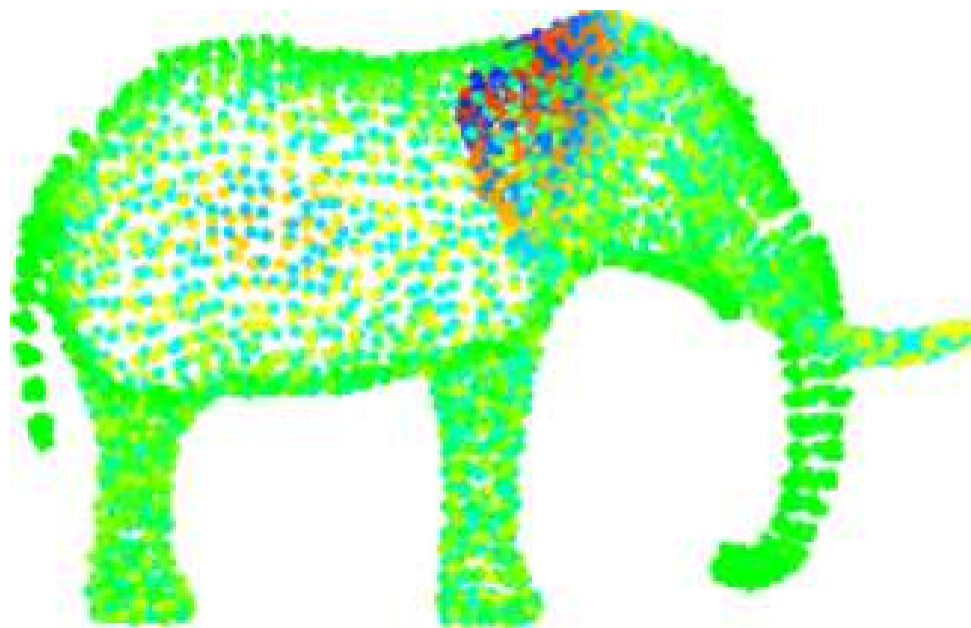
coords from $\omega/\text{Jeff2207}$

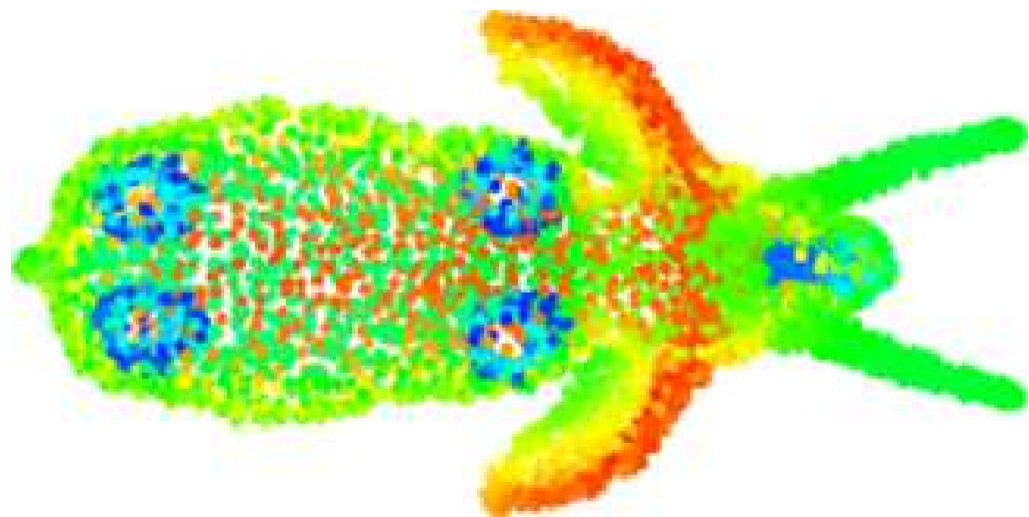


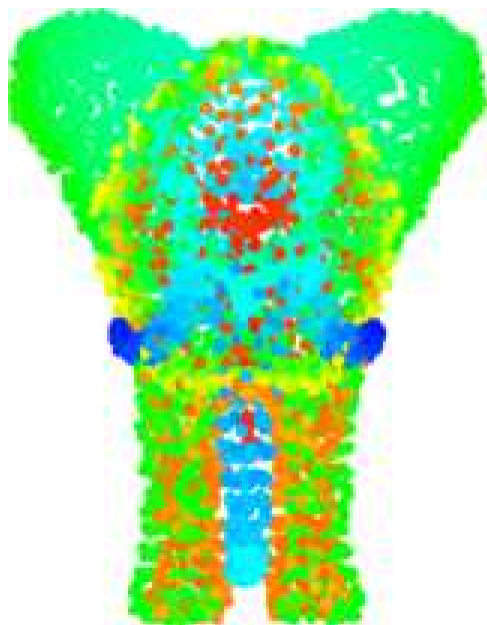














coords from [ω/Jeff2207](#)



ω/g



ω/r



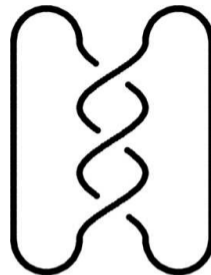
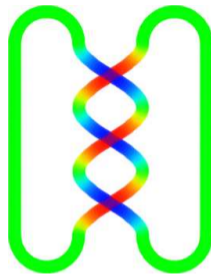
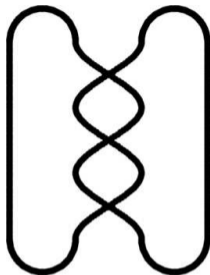
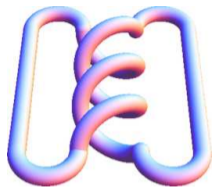
ω/b



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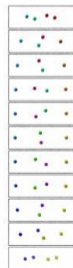
Knots.



“broken curve diagram”

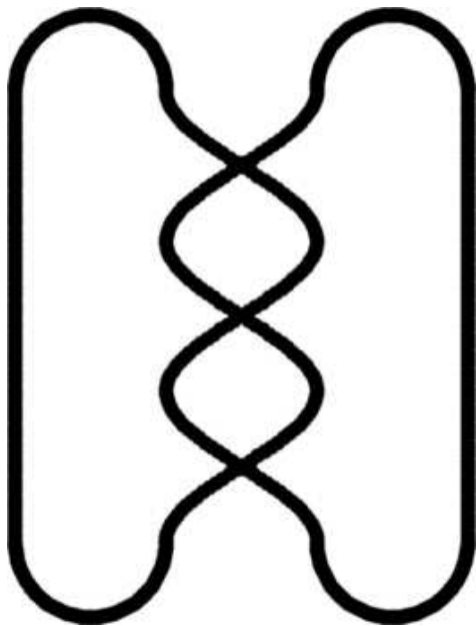


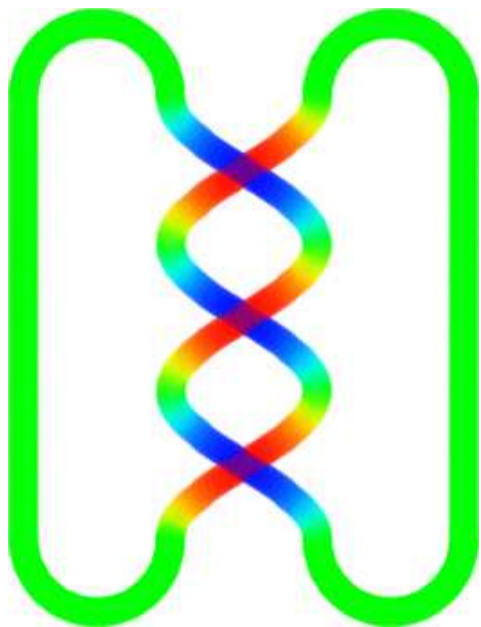
with Ester Dalvit ω /Dal

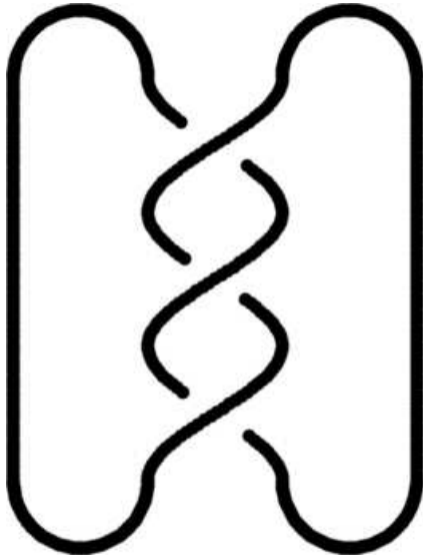


ω /M2

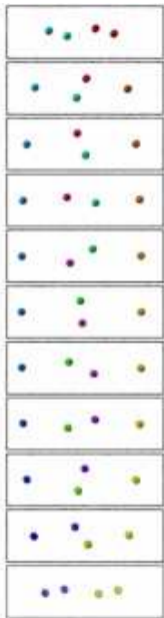
Formally, “a differentiable embedding of S^1 in \mathbb{R}^3 modulo differentiable deformations of such”.







“broken curve diagram”



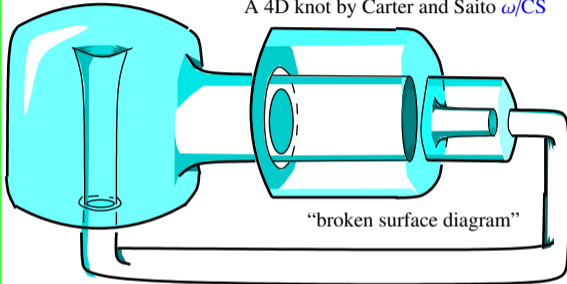
$\omega/M2$

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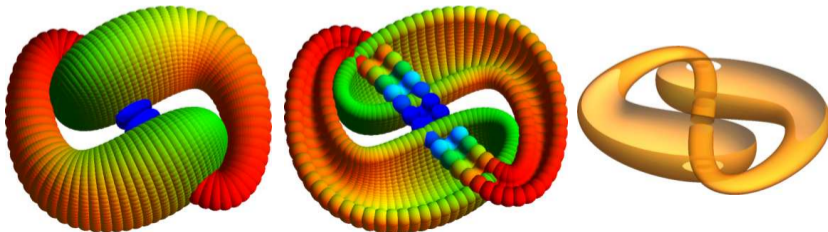
2-Knots / 4D Knots.

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A 4D knot by Carter and Saito ω/CS

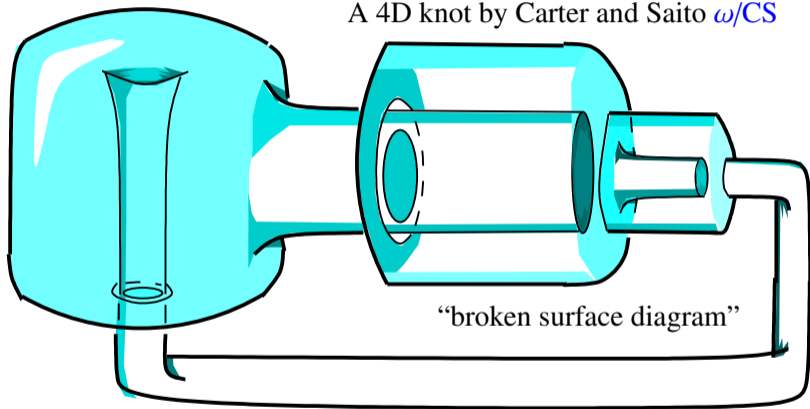


Carter, Banach, Saito

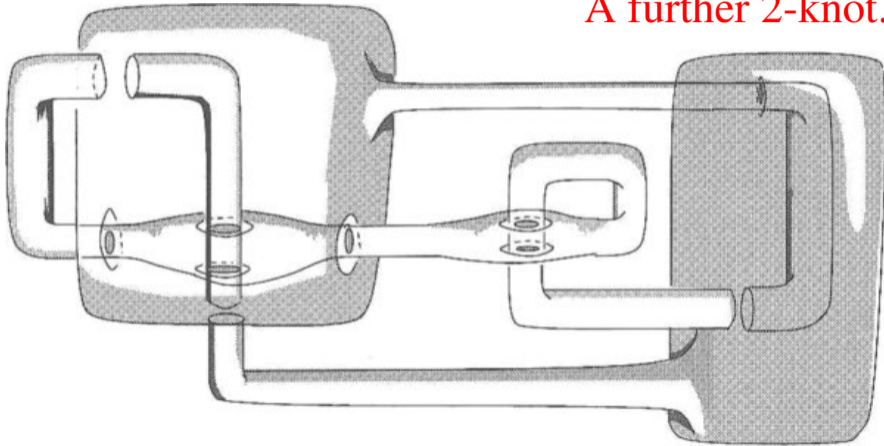


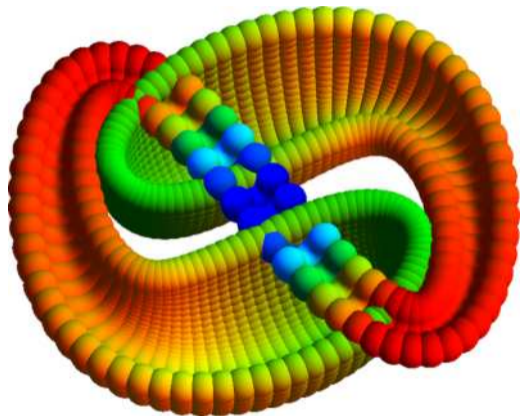
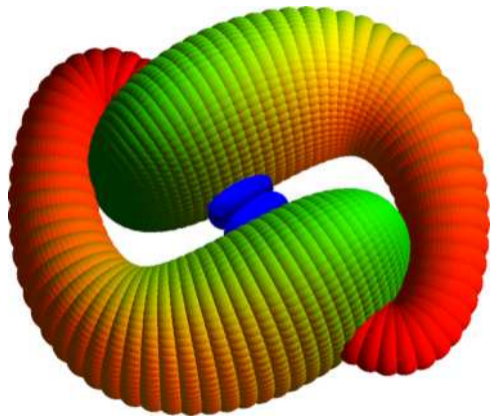
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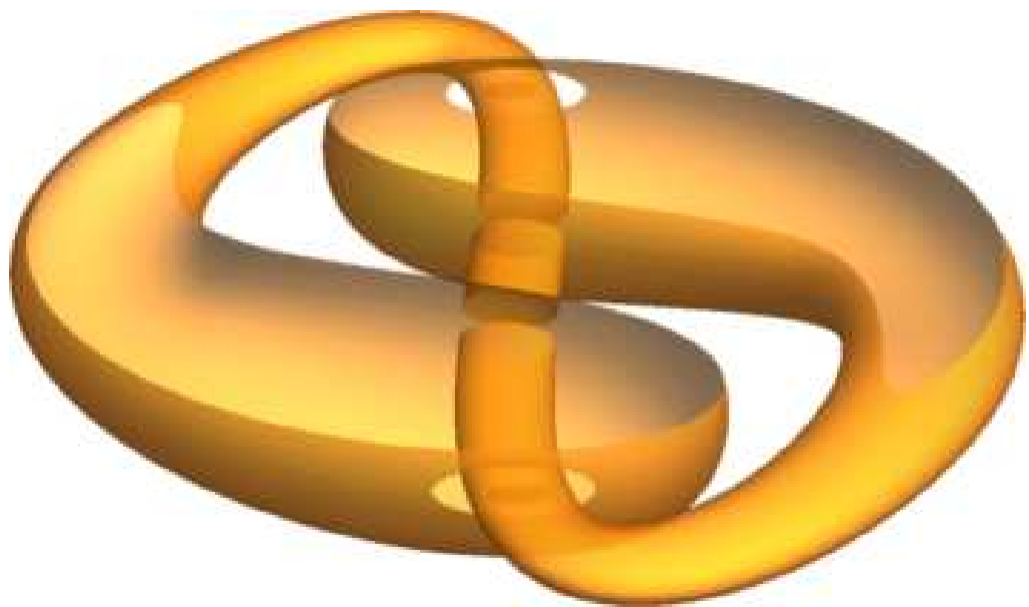
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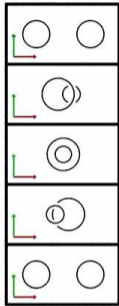
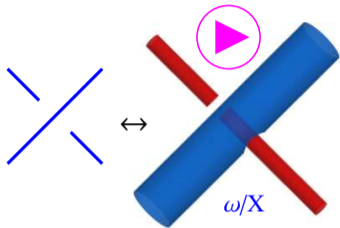
A further 2-knot.



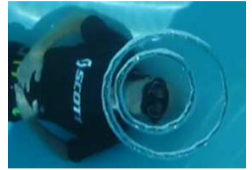
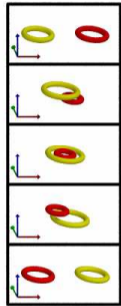




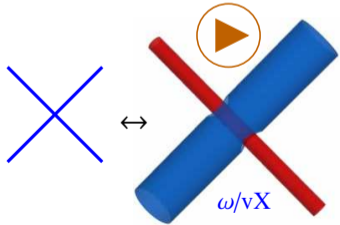
Some Movies



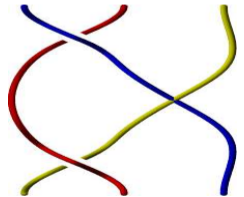
$\omega/X1$

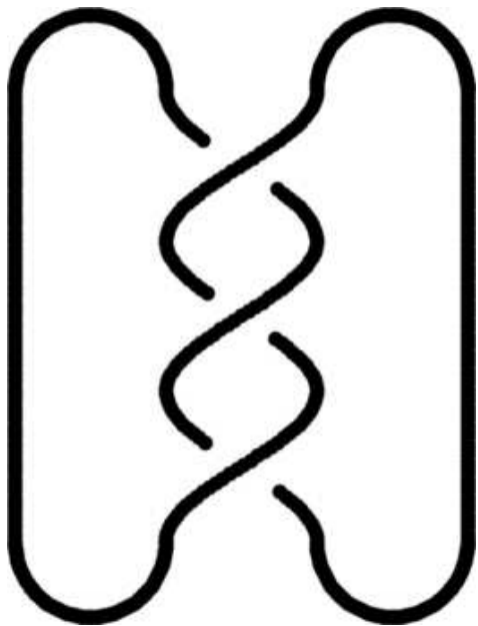


ω/Bub

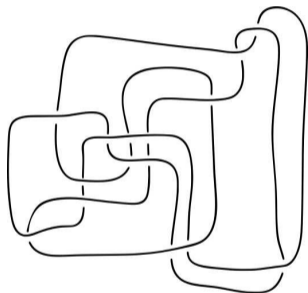


ω/F





Some Unknots



Thistlethwaite's unknot



ω/U

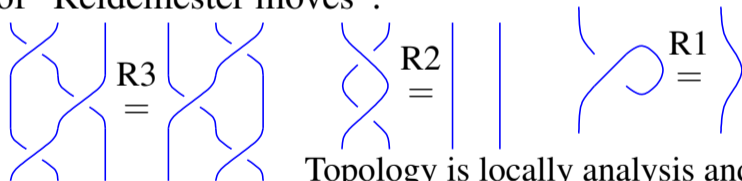


Scharein's relaxation



Haken's unknot

Reidemeister's Theorem. (a) Every knot has a “broken curve diagram”, made only of curves and “crossings” like \times . (b) Two knot diagrams represent the same 3D knot iff they differ by a sequence of “Reidemeister moves”:



Kurt Reidemeister

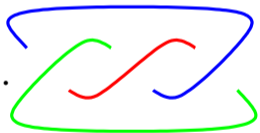
Topology is locally analysis and globally algebra

3-Colourings. Colour the arcs of a broken arc diagram in **RGB** so that every crossing is either mono-chromatic or tri-chromatic. Let $\lambda(K)$ be the number of such 3-colourings that K has.

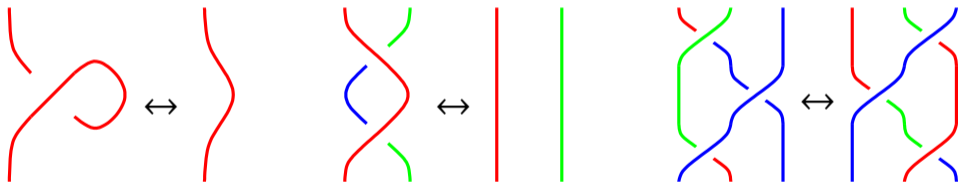


Example. $\lambda(\bigcirc) = 3$ while $\lambda(\text{Ⓞ}) = 9$; so $\bigcirc \neq \text{Ⓞ}$.

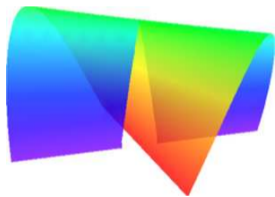
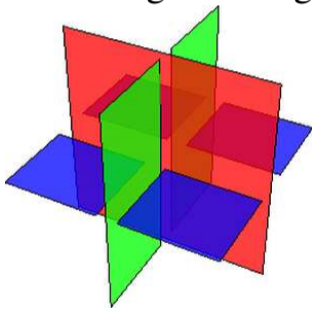
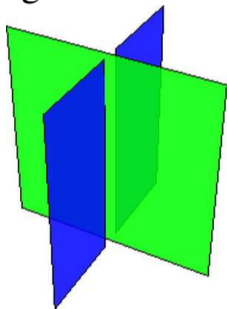
Riddle. Is $\lambda(K)$ always a power of 3?



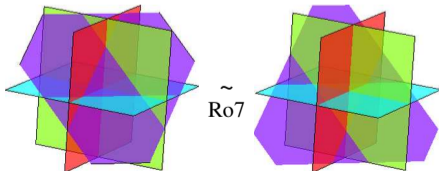
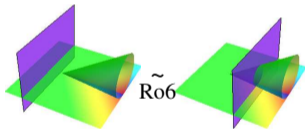
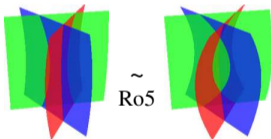
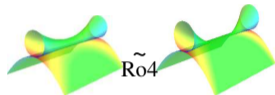
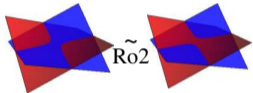
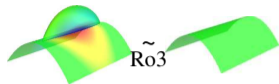
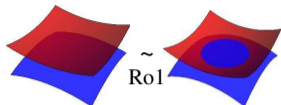
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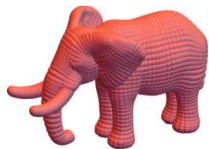
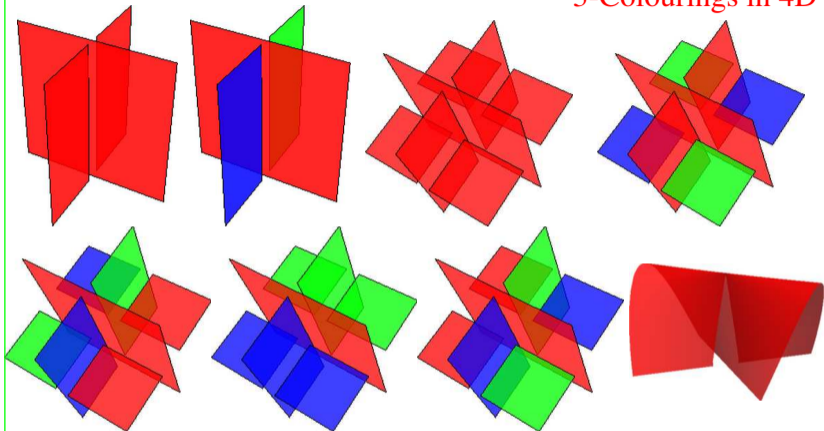


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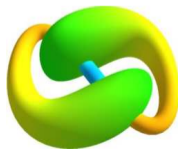


D. Roseman

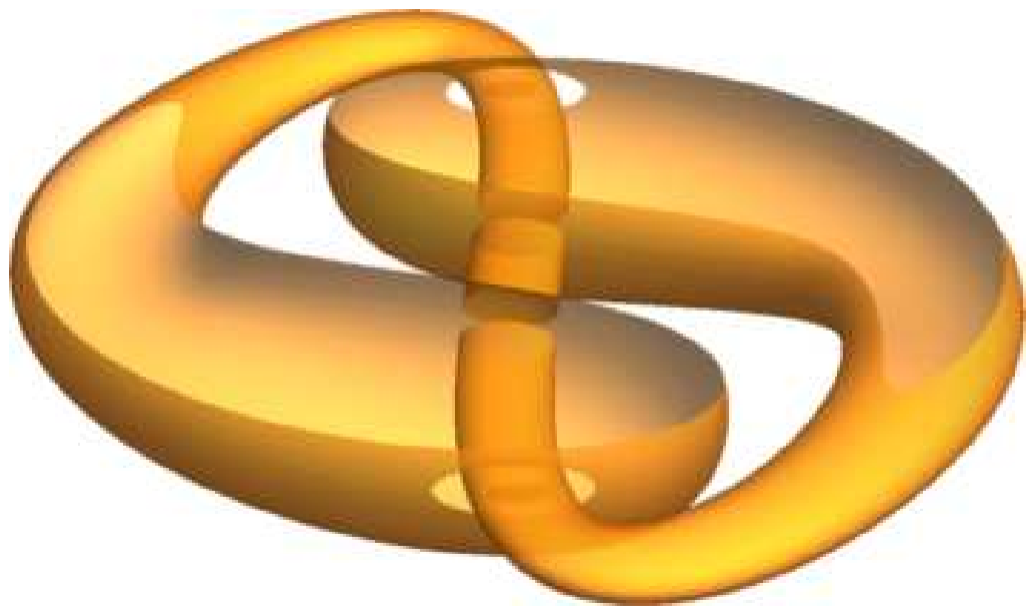
3-Colourings in 4D



→ 3

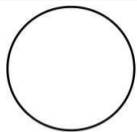


→ 9



A Knot Table

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more!



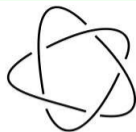
Unknot



3_1



4_1



5_1



5_2



6_1



6_2



6_3



7_1



7_2



7_3



7_4



7_5



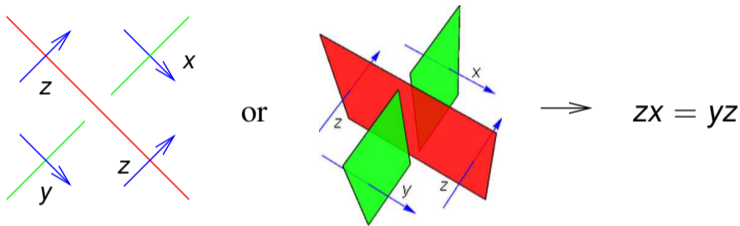
7_6



7_7

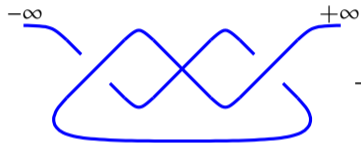
ω/KT

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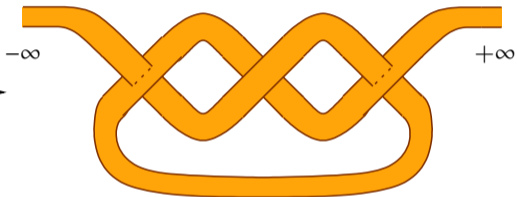


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“long w-knot diagram”

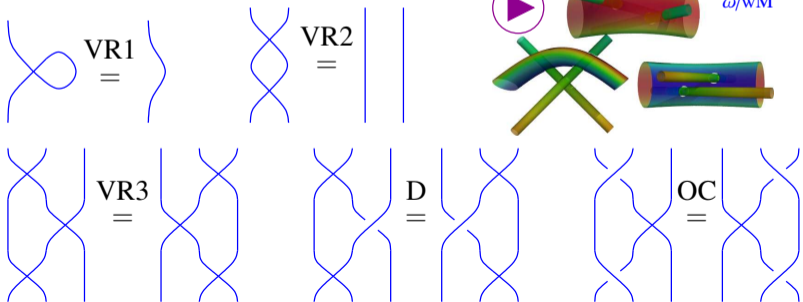


→ “simple long knotted 2D tube in 4D”

Satoh's Conjecture. (Satoh, *Virtual Knot Presentations of Ribbon Torus-Knots*, *J. Knot Theory and its Ramifications* **9** (2000) 531–542). Two long w-knot diagrams represent via the map δ the same simple long 2D knotted tube in 4D iff they differ by a sequence of R-moves as above and the “w-moves” VR1–VR3, D and OC listed below:



Shin Satoh



Some knot theory books.

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