Dogma handout on 170803

August 3, 2017 8:43 AM

update

update.

Dror Bar-Natan: Talks: Toulouse-1705:

The Dogma is Wrong

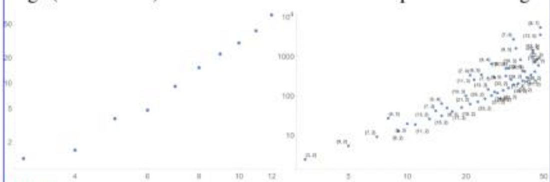
Follows Rozansky [Ro1, Ro2, Ro3] and Overbay [Ov], joint with van der Veen. More at ωεβ/talks.

Thanks for the invitation! ωεβ:=http://drorbn.net/Toulouse-1705/

Abstract. It has long been known that there are knot invariants associated to semi-simple Lie algebras, and there has long been a dogma as for how to extract them: "quantize and use representation theory". We present an alternative and better procedure: "centrally extend, approximate by solvable, and learn how to re-order exponentials in a universal enveloping algebra". While equivalent to the old invariants via a complicated process, our invariants are in practice stronger, faster to compute (poly-time vs. exp-time), and clearly carry topological information.

KiW 43 Abstract (ωεβ/kiw). Whether or not you like the formulas on this page, they describe the strongest truly computable knot invariant we know.

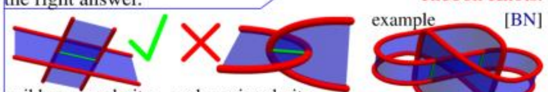
Experimental Analysis (ωεβ/Exp). Log-log plots of computation time (sec) vs. crossing number, for all knots with up to 12 crossings (mean times) and for all torus knots with up to 48 crossings:



Power. On the 250 knots with at most 10 crossings, the pair (ω, ρ_1) attains 250 distinct values, while (Khovanov, HOMFLY-PT) attains only 249 distinct values. To 11 crossings the numbers are (802, 788, 772) and to 12 they are (2978, 2883, 2786).

Genus. Up to 12 xings, always ρ_1 is symmetric under $t \leftrightarrow t^{-1}$. With ρ_1^+ denoting the positive-degree part of ρ_1, always $\deg \rho_1^+ \leq 2g-1$, where g is the 3-genus of K (equality for 2530 knots). This gives a lower bound on g in terms of ρ_1 (conjectural, but undoubtedly true). This bound is often weaker than the Alexander bound, yet for 10 of the 12-xing Alexander failures it does give the right answer.

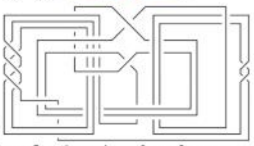
Ribbon Knots.



a ribbon singularity a clasp singularity

Gompf, Scharlemann, Thompson [GST]

$$\begin{array}{ccc} & U \in \mathcal{T}_n & & 1 \in \mathcal{A}_n & \\ & \tau \nearrow & & \tau\nearrow & \\ \mathcal{T}_{2n} & & \xrightarrow{z} & \mathcal{A}_{2n} & \\ & \kappa \searrow & & \kappa\searrow & \\ & \text{ribbon } K \in \mathcal{T}_1 & & z(K) \in \mathcal{R} \subseteq \mathcal{A}_1 & \end{array}$$



[Vo]: Works for Alexander!  with $\mathcal{R} := \kappa(\tau^{-1}(1))$

$A^+ = -t^8 + 2t^7 - t^6 - 2t^4 + 5t^3 - 2t^2 - 7t + 13$

$\rho_1^+ = 5t^{15} - 18t^{14} + 33t^{13} - 32t^{12} + 2t^{11} + 42t^{10} - 62t^9 - 8t^8 + 166t^7 - 242t^6 + 108t^5 + 132t^4 - 226t^3 + 148t^2 - 11t - 36$

Faster is better, leaner is meaner!

dog·ma (dôg′mə, dŏg′-) The Free Dictionary, ωεβ/TFD

n. pl. **dog·mas** or **dog·ma·ta** (-mə-tə)

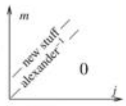
1. A doctrine or a corpus of doctrines relating to matters such as morality and faith, set forth in an authoritative manner by a religion.
2. A principle or statement of ideas, or a group of such principles or statements, especially when considered to be authoritative or accepted uncritically: *"Much education consists in the instilling of unfounded dogmas in place of a spirit of inquiry" (Bertrand Russell).*

Theorem ([BNG], conjectured [MM], elucidated [Ro1]). Let $J_d(K)$ be the coloured Jones polynomial of K, in the d-dimensional representation of sl_2. Writing

Melvin, Morton, Garoufalidis

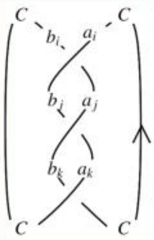
$$\left.\frac{(q^{1/2} - q^{-1/2})J_d(K)}{q^{d/2} - q^{-d/2}}\right|_{q=e^\hbar} = \sum_{j,m\geq 0} a_{jm}(K) d^j \hbar^m,$$

"below diagonal" coefficients vanish, $a_{jm}(K) = 0$ if $j > m$, and "on diagonal" coefficients give the inverse of the Alexander polynomial: $\left(\sum_{m=0}^\infty a_{mm}(K)\hbar^m\right) \cdot \omega(K)(e^\hbar) = 1$.



"Above diagonal" we have Rozansky's Theorem [Ro3, (1.2)]:

$$J_d(K)(q) = \frac{q^d - q^{-d}}{(q-q^{-1})\omega(K)(q^d)} \left(1 + \sum_{k=1}^\infty \frac{(q-1)^k \rho_k(K)(q^d)}{\omega^{2k}(K)(q^d)}\right).$$



The Yang-Baxter Technique. Given an algebra A (typically $\hat{\mathcal{U}}(\mathfrak{g})$ or $\hat{\mathcal{U}}_q(\mathfrak{g})$) and elements

$$R = \sum a_i \otimes b_i \in A \otimes A \quad \text{and} \quad C \in A,$$

form

$$Z = \sum_{i,j,k} C a_i b_j a_k C^2 b_i a_j b_k C.$$

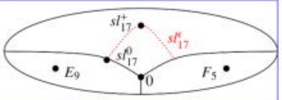
Problem. Extract information from Z.

The Dogma. Use representation theory. In principle finite, but *slow*.

The Loyal Opposition. For certain algebras, work in a homomorphic poly-dimensional "space of formulas".

$$m_k^{ij} \circlearrowright \{\mathcal{F}_S\} \xrightarrow{\mathbb{E}} \{A^{\otimes S}\} \circlearrowleft m_k^{ij}$$

The (fake) moduli of Lie algebras on V, a quadratic variety in $(V^*)^{\otimes 2} \otimes V$ is on the right. We care about $sl_{17}^k := sl_{17}^\epsilon / (\epsilon^{k+1} = 0)$.



use pic from PPSA paper

A

Why are "solvable algebras" any good? Contrary to common beliefs, computations in semi-simple Lie algebras are just awful:

```
In[1]:= MatrixExp[( a b
                    c d )] // FullSimplify // MatrixForm      Enter
```

Yet in solvable algebras, exponentiation is fine and even BCH, $z = \log(e^x e^y)$, is bearable:

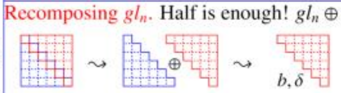
```
In[2]:= MatrixExp[( a b
                    0 c )] // MatrixForm
```

Out[2]//MatrixForm=

$$\begin{pmatrix} e^a & \frac{b(e^a - e^c)}{a-c} \\ 0 & e^c \end{pmatrix}$$

```
In[3]:= MatrixExp[( a1 b1
                    0  c1 )].MatrixExp[( a2 b2
                                         0  c2 )] //
          MatrixLog // PowerExpand // Simplify //
        MatrixForm                                        Enter
```

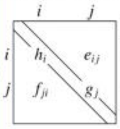
Recomposing gl_n. Half is enough! $gl_n \oplus \mathfrak{a}_n = \mathcal{D}(\triangledown, b, \delta)$:



$b(\triangledown) = b: \triangledown \otimes \triangledown \to \triangledown$

$b(\triangle) \rightsquigarrow \delta: \triangledown \to \triangledown \otimes \triangledown$

Now define $gl_n^\epsilon := \mathcal{D}(\triangledown, b, \epsilon\delta)$. Schematically, this is $[\triangledown, \triangledown] = \triangledown$, $[\triangle, \triangle] = \epsilon\triangle$, and $[\triangledown, \triangle] = \triangle + \epsilon\triangledown$. In detail, it is



$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{li} e_{kj} \qquad [f_{ij}, f_{kl}] = \epsilon\delta_{jk} f_{il} - \epsilon\delta_{li} f_{kj}$$

$$[e_{ij}, f_{kl}] = \delta_{jk}(\epsilon\delta_{j<k} e_{il} + \delta_{il}(h_i + \epsilon g_i)/2 + \delta_{i>l} f_{il}) - \delta_{li}(\epsilon\delta_{k<j} e_{kj} + \delta_{kj}(h_j + \epsilon g_j)/2 + \delta_{k>j} f_{kj})$$

$$[g_i, e_{jk}] = (\delta_{ij} - \delta_{ik}) e_{jk} \qquad [h_i, e_{jk}] = \epsilon(\delta_{ij} - \delta_{ik}) e_{jk}$$

$$[g_i, f_{jk}] = (\delta_{ij} - \delta_{ik}) f_{jk} \qquad [h_i, f_{jk}] = \epsilon(\delta_{ij} - \delta_{ik}) f_{jk}$$

A: Another Dogma: semi-simple Lie algebras are the centre of the universe.

The (weaker) loyal opposition: No, the $\mathfrak{g}_0$'s are still centre.

The sl_2 Example. Let $\mathfrak{g}^\epsilon = \langle h,e,l,f\rangle/([h,\cdot]=0,\,[e,l]=-e,\,[f,l]=f,\,[e,f]=h-2\epsilon l)$ and let $\mathfrak{g}_k=\mathfrak{g}^\epsilon/(\epsilon^{k+1}=0)$.

The Main $\mathfrak{g}_k$ Theorem. The $\mathfrak{g}_k$-invariant of any S-component tangle T can be written in the form

$$Z(T)=\mathbb{O}\left(\omega e^{L+Q+P}:\bigotimes_{i\in S} e_i l_i f_i\right),$$

where ω is a scalar (meaning, a rational function in the variables h_i and their exponentials $t_i:=e^{h_i}$), where $L=\sum a_{ij}h_i l_j$ is a balanced quadratic in the variables h_i and l_j with integer coefficients, where $Q=\sum b_{ij}e_i f_j$ is a balanced quadratic in the variables e_i and f_j with scalar coefficients b_{ij}, and where P is a polynomial in $\{\epsilon,e_i,l_i,f_i\}$ (with scalar coefficients) whose ϵ^d-term is of degree at most $2d+2$ in $\{e_i,\sqrt{l_i},f_i\}$. Furthermore, after setting $h_i=h$ and $t_i=t$ for all i, the invariant $Z(T)$ is poly-time computable.

The Main $\mathfrak{g}_k$ Lemma. The following "re-ordering relations" hold:

$$\mathbb{O}\left(e^{\gamma l+\beta e}: le\right)=\mathbb{O}\left(e^{\gamma l+e^\gamma\beta e}: el\right)\quad\text{(and similarly for } fl\to lf),$$

$$\mathbb{O}\left(e^{\beta e+\alpha f+\delta ef}: fe\right)=\mathbb{O}\left(\nu e^{\nu(-\alpha\beta h+\beta e+\alpha f+\delta ef)+\lambda_k(\epsilon,e,l,f,\alpha,\beta,\delta)}: elf\right),$$

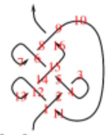
with $\nu=(1+h\delta)^{-1}$ and where $\lambda_k(\epsilon,e,l,f,\alpha,\beta,\delta)$ is some fixed polynomial of degree at most $2k+2$ in $\epsilon,e,\sqrt{l},f,\alpha,\beta,\delta$, with scalar coefficients.

Demo Programs. ωεβ/Demo

Formatting
```
CF[ℰ_] := Module[{vars = Union@Cases[ℰ, e_ | l_ | f_, ∞]},
  If[vars === {}, Factor[ℰ],
   Total[CoefficientRules[ℰ, vars] /.
     (p_ → c_) ⧴ Factor[c] Times @@ (vars^p)] ]];
CF[ℰ_𝔼] := CF /@ ℰ;
```

Preparation
```
𝔼[i_, j_, s_] := 𝔼[1, (-1)^s l_j, (-t)^s e_i f_j,
  t^s e_i l_(1+s) i-s j f_j + (-1)^s l_i l_j + (-t^2)^s e_i^2 f_j^2/4];
𝔼[i_, s_] := 𝔼[1, 0, 0, s l_i];
𝔼 /: 𝔼[1, L1_, Q1_, P1_] 𝔼[1, L2_, Q2_, P2_] :=
  𝔼[1, L1 + L2, Q1 + Q2, P1 + P2];
```

Preparing the Trefoil
```
z1 = (𝔼[1, 11, 0] 𝔼[4, 2, -1] 𝔼[15, 5, 0] ×
   𝔼[6, 8, -1] 𝔼[9, 16, 0] 𝔼[12, 14, -1] ×
   𝔼[3, -1] 𝔼[7, +1] 𝔼[10, -1] 𝔼[13, +1])
```



$$\mathbb{E}\Big[1,\ -l_2+l_5-l_8+l_{11}-l_{14}+l_{16},\ -\tfrac{e_4 f_2}{t}+e_{15}f_5-\tfrac{e_6 f_8}{t}+e_1 f_{11}-\tfrac{e_{12}f_{14}}{t}+e_9 f_{16},$$
$$-\tfrac{e_4^2 f_2^2}{4t^2}+\tfrac14 e_{15}^2 f_5^2-\tfrac{e_6^2 f_8^2}{4t^2}+\tfrac14 e_1^2 f_{11}^2-\tfrac{e_{12}^2 f_{14}^2}{4t^2}+\tfrac14 e_9^2 f_{16}^2+e_1 f_{11} l_1+$$
$$\tfrac{e_4 f_2 l_2}{t}-l_3-l_2 l_4+l_7+\tfrac{e_6 f_8 l_8}{t}-l_6 l_8+e_9 f_{16} l_9-l_{10}+$$
$$l_1 l_{11}+l_{13}+\tfrac{e_{12}f_{14}l_{14}}{t}-l_{12}l_{14}+e_{15}f_5 l_{15}+l_5 l_{15}+l_9 l_{16}\Big]$$

Differential Polynomials (Implementing $P(\partial_\alpha,\partial_\beta)(f)$)
```
DP_{x_→D_α_, y_→D_β_}[P_][f_] :=
 Total[CoefficientRules[P, {x, y}] /.
   ({m_, n_} → c_) ⧴ c D[f, {α, m}, {β, n}]]
```

le and *fl* Sorts
```
S_{l_j_ (x:e|f)_i_ → k_}[𝔼[ω_, L_, Q_, P_]] :=
 With[{λ = ∂_{l_j} L, α = ∂_{x_i} Q, q = e^γ β x_k + γ l_k}, CF[
   𝔼[ω, L /. l_j → l_k, t^λ α x_k + (Q /. x_i → 0),
    e^-q DP_{l_j→D_γ, x_i→D_β}[P][e^q] /. {β → α/ω, γ → λ Log[t]}] ]];
```

The Λόγος
```
Λ[k_] := ((t - 1) (2 (α β + δ μ)^2 - α^2 β^2) - 4 e_k l_k f_k δ^2 μ^2 -
    δ (1 + μ) (f_k^2 α^2 + e_k^2 β^2) - e_k^2 f_k^2 δ^3 (1 + 3 μ) -
    2 (α β + 2 δ μ + e_k f_k δ^2 (1 + 2 μ) + 2 l_k δ μ^2) (f_k α + e_k β) -
    4 (l_k μ^2 + e_k f_k δ (1 + μ)) (α β + δ μ)) (1 + t) / 4;
```

fe Sorts
```
S_{f_i_ e_j_ → k_}[𝔼[ω_, L_, Q_, P_]] :=
 With[{q = ((1 - t) α β + β e_k + α f_k + δ e_k f_k) / μ}, CF[
   𝔼[μ ω, L, μ ω q + μ (Q /. f_i | e_j → 0),
     μ^4 e^-q DP_{f_i→D_α, e_j→D_β}[P][e^q] + ω^4 Λ[k]] /. μ → 1 + (t - 1) δ /.
   {α → ω^-1 (∂_{f_i} Q /. e_j → 0), β → ω^-1 (∂_{e_j} Q /. f_i → 0),
    δ → ω^-1 ∂_{f_i,e_j} Q}]];
```

Elf Merges
```
m_{i_,j_→k_}[Z_𝔼] := Module[{x, z},
  CF[(Z // S_{f_i e_j→x} // S_{l_i e_x→x} // S_{f_x l_j→x}) /. z_i|j|x → z_k]]
```

Rewriting the Trefoil (by merging 16 elves)
```
(Do[z1 = z1 // m_{1,k→1}, {k, 2, 16}]; z1)
```

$$\mathbb{E}\Big[\tfrac{1-t+t^2}{t},\ 0,\ 0,\ \tfrac{(-1+t)(1-t+t^2)^2(1-t+2t^2)}{t^3}-\tfrac{2(1+t)(1-t+t^2)^3 e_1 f_1}{t^4}-\tfrac{2(-1+t)(1+t)(1-t+t^2)^3 l_1}{t^4}\Big]$$

Readout
```
ρ_1[𝔼[ω_, _, _, P_]] := CF[ t ((P /. e_ | l_ | f_ → 0) - t ω^3 (∂_t ω)) / ((t - 1)^2 ω^2) ]
```

$\rho_1(3_1)$
```
ρ_1[z1] // Expand
```

$\frac{1}{t}+t$

References.

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[Ro2] L. Rozansky, *The Universal R-Matrix, Burau Representation and the Melvin-Morton Expansion of the Colored Jones Polynomial,* Adv. Math. **134-1** (1998) 1–31, arXiv:q-alg/9604005.

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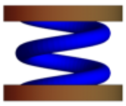
[Vo] H. Vo, University of Toronto Ph.D. thesis, in preparation.



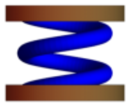
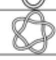


diagram	n^t_k Alexander's ω^+ / Today's / Rozansky's ρ_1^+	genus / ribbon / unknotting number / amphicheiral	diagram	n^t_k Alexander's ω^+ / Today's / Rozansky's ρ_1^+	genus / ribbon / unknotting number / amphicheiral
	0^a_1 1 0	0 / ✔ 0 / ✔		3^a_1 $t-1$ t	1 / ✘ 1 / ✘
	4^a_1 $3-t$ 0	1 / ✘ 1 / ✔		5^a_1 t^2-t+1 $2t^3+3t$	2 / ✘ 2 / ✘

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