Dror Bar - Natan: Talks: Sydney-1708:

Nobody Solves the Quintic

University of Sydney Undergraduate Lecture, August 2017

Abstract. Everybody knows that nobody can solve the quintic. Indeed this insolubility is a well known hard theorem, the high point of a full-semester course on Galois theory, often taken in one's 3rd or 4th year of university mathematics. I'm not sure why so few know that the same theorem can be proven in about 15 minutes using *very* basic and easily understandable topology, accessible to practically anyone.

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Handout

http://www.math.toronto.edu/~drorbn/Talks/Sydney-1708/
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Abstract. Everybody knows that nobody can solve the quintic. Indeed this insolubility is a well known hard theorem, the high point of a full-semester course on Galois theory, often taken in one's 3rd or 4th year of university mathematics. I'm not sure why so few know that the same theorem can be proven in about 15 minutes using *very* basic and easily understandable topology, accessible to practically anyone.

Definition. The commutator of two elements x and y in a group G is $[x,y] := xyx^{-1}y^{-1}$.

Example 0. In $\mathbb{Z}$, $[m,n]=0$.

Example 1. In S_3, $[(12),(23)] = (12)(23)(12)^{-1}(23)^{-1} = (123)$ and in general in $S_{\geq 3}$,

$$[(ij),(jk)] = (ijk).$$

Example 2. In $S_{\geq 4}$,

$$[(ijk),(jkl)] = (ijk)(jkl)(ijk)^{-1}(jkl)^{-1} = (il)(jk).$$

Example 3. In $S_{\geq 5}$,

$$[(ijk),(klm)] = (ijk)(klm)(ijk)^{-1}(klm)^{-1} = (jkm).$$

Example 4. So, in fact, in S_5, $(123) = [(412),(253)] = [[(341),(152)],[(125),(543)]] = [[[(234),(451)],[(315),(542)]],[[(312),(245)],[(154),(423)]]] = [[[[(123),(354)],[(245),(531)]],[[(231),(145)],[(154),(432)]]],[[[(431),(152)],[(124),(435)]],[[(215),(534)],[(142),(253)]]]]$.

Solving the Quadratic, $ax^2+bx+c=0$: $\delta = \sqrt{\Delta}$; $\Delta = b^2-4ac$; $r=\frac{\delta-b}{2a}$.

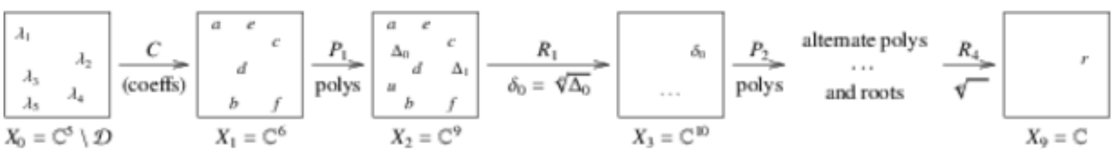
Solving the Cubic, $ax^3+bx^2+cx+d=0$: $\Delta = 27a^2d^2-18abcd+4ac^3+4b^3d-b^2c^2$; $\delta = \sqrt{\Delta}$; $\Gamma = 27a^2d-9abc+3\sqrt{3}a\delta+2b^3$; $\gamma = \sqrt[3]{\frac{\Gamma}{2}}$; $r = -\frac{\frac{b^2-3ac}{\gamma}+b+\gamma}{3a}$.

Solving the Quartic, $ax^4+bx^3+cx^2+dx+e=0$: $\Delta_0 = 12ae-3bd+c^2$; $\Delta_1 = -72ace+27ad^2+27b^2e-9bcd+2c^3$; $\Delta_2 = \frac{1}{27}\left(\Delta_1^2-4\Delta_0^3\right)$; $u = \frac{8ac-3b^2}{8a^2}$; $v = \frac{8a^2d-4abc+b^3}{8a^3}$; $\delta_2 = \sqrt{\Delta_2}$; $Q = \frac{1}{2}\left(3\sqrt{3}\delta_2+\Delta_1\right)$; $q = \sqrt[3]{Q}$; $S = \frac{\frac{\Delta_0}{q}+q}{12a} - \frac{u}{6}$; $s = \sqrt{S}$; $\Gamma = -\frac{v}{s}-4S-2u$; $\gamma = \sqrt{\Gamma}$; $r = -\frac{b}{4a}+\frac{\gamma}{2}+s$.

Theorem. The is no general formula, using only the basic arithmetic operations and taking roots, for the solution of the quintic equation $ax^5+bx^4+cx^3+dx^2+ex+f=0$.

Key Point. The "persistent root" of a closed path (path lift, in topological language) may not be closed, yet the persistent root of a commutators of closed paths is always closed.

Proof. Suppose there was a formula, and consider the corresponding "composition of machines" picture:



Now if $\gamma_1^{(1)}, \gamma_2^{(1)}, \ldots, \gamma_{16}^{(1)}$, are "musical chairs" paths in X_0 that induce permutations of the roots and we set $\gamma_1^{(2)} := [\gamma_1^{(1)},\gamma_2^{(1)}]$, $\gamma_2^{(2)} := [\gamma_3^{(1)},\gamma_4^{(1)}], \ldots, \gamma_8^{(2)} := [\gamma_{15}^{(1)},\gamma_{16}^{(1)}]$, $\gamma_1^{(3)} := [\gamma_1^{(2)},\gamma_2^{(2)}], \ldots, \gamma_4^{(3)} := [\gamma_7^{(2)},\gamma_8^{(2)}]$, $\gamma_1^{(4)} := [\gamma_1^{(3)},\gamma_2^{(3)}]$, $\gamma_2^{(4)} := [\gamma_3^{(3)},\gamma_4^{(3)}]$, and finally $\gamma^{(5)} := [\gamma_1^{(4)},\gamma_2^{(4)}]$ (notes: (1) these commutators make sense! (2) all of those are commutators of "long paths" (3) I don't know the word "homotopy"), then $\gamma^{(5)} /\!\!/ C /\!\!/ P_1 /\!\!/ R_1 /\!\!/ \cdots /\!\!/ R_4$ is a closed path. Indeed,

- In X_0, none of the paths is necessarily closed.
- After C, all of the paths are closed.
- After P_1, all of the paths are still closed.
- After R_1, the $\gamma^{(1)}$'s may open up, but the $\gamma^{(2)}$'s remain closed.

…

- At the end, after R_4, $\gamma^{(4)}$'s may open up, but $\gamma^{(5)}$ remains closed.


V.I. Arnold

But if the paths are chosen as in Example 4, $\gamma^{(5)} /\!\!/ C /\!\!/ P_1 /\!\!/ R_1 /\!\!/ \cdots /\!\!/ R_4$ is not a closed path. □

References. V.I. Arnold, 1960s, hard to locate.
V.B. Alekseev, *Abel's Theorem in Problems and Solutions, Based on the Lecture of Professor V.I. Arnold*, Kluwer 2004.
A. Khovanskii, *Topological Galois Theory, Solvability and Unsolvability of Equations in Finite Terms*, Springer 2014.
B. Katz, *Short Proof of Abel's Theorem that 5th Degree Polynomial Equations Cannot be Solved*, YouTube video, http://youtu.be/RhpVSV6iCko.

Definitions and Very Simple Examples

Definition. The commutator of two operations A and B is $[A, B] := ABA^{-1}B^{-1}$, or "do A, do B, undo A, undo B".

Example 0. In $\mathbb{Z}$, $[m, n] = 0$.

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Example 1. In S_3, $[(12), (23)] = (12)(23)(12)^{-1}(23)^{-1} = (123)$ and in general in $S_{\geq 3}$, [(ij),(jk)]=(ijk).

Example 2. In $S_{\geq 4}$, $[(ijk), (jkl)] = (ijk)(jkl)(ijk)^{-1}(jkl)^{-1} = (il)(jk)$.

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].

The Quintic

$$ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$$

Can you solve the quintic in radicals? Is there a formula for the zeros of a degree 5 polynomial in terms of its coefficients, using only the operations on a scientific calculator? $\left(+, -, \times, \div, \sqrt[n]{a}\right)$

History: First solved by Abel / Galois in the 1800s. Our solution follows Arnold’s topological solution from the 1960s. I could not find the original writeup by Arnold (if it at all exists), yet see:

V.B. Alekseev, *Abel’s Theorem in Problems and Solutions, Based on the Lecture of Professor V.I. Arnold,* Kluwer 2004.

A. Khovanskii, *Topological Galois Theory, Solvability and Unsolvability of Equations in Finite Terms,* Springer 2014.

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Solving the Quadratic $ax^2 + bx + c = 0$

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Δ = b^2 - 4 a c;
δ = √Δ;
r = (-b + δ) / (2 a)
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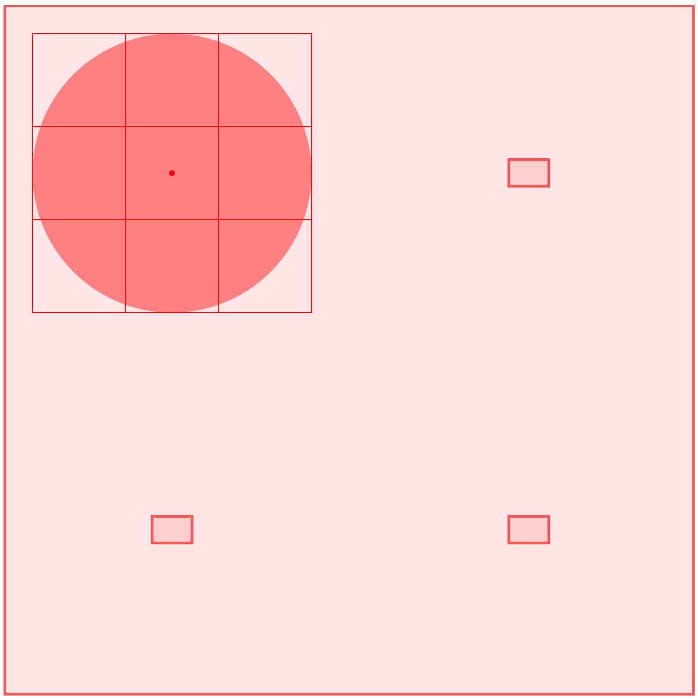
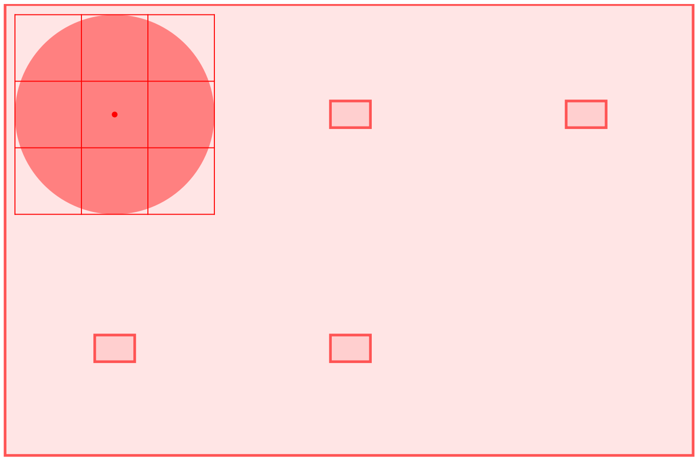
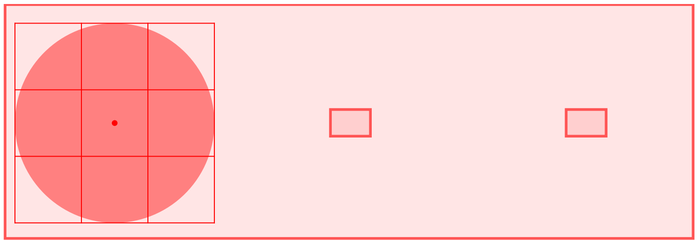


Testing the Quadratic Solution



Square Roots and Persistent Square Roots



Leading Questions

“Yes, Prime Minister”, 1986.

Sir Humphrey: You know what happens: nice young lady comes up to you. Obviously you want to create a good impression, you don’t want to look a fool, do you? So she starts asking you some questions: Mr. Woolley, are you worried about the number of young people without jobs?
Bernard Woolley: Yes
Sir Humphrey: Are you worried about the rise in crime among teenagers?
Bernard Woolley: Yes
Sir Humphrey: Do you think there is a lack of discipline in our Comprehensive schools?
Bernard Woolley: Yes
Sir Humphrey: Do you think young people welcome some authority and leadership in their lives?
Bernard Woolley: Yes
Sir Humphrey: Do you think they respond to a challenge?
Bernard Woolley: Yes
Sir Humphrey: Would you be in favour of reintroducing National Service?
Bernard Woolley: Oh...well, I suppose I might be.
Sir Humphrey: Yes or no?
Bernard Woolley: Yes
Sir Humphrey: Of course you would, Bernard. After all you told me can’t say no to that. So they don’t mention the first five questions and they publish the last one.
Bernard Woolley: Is that really what they do?
Sir Humphrey: Well, not the reputable ones no, but there aren’t many of those. So alternatively the young lady can get the opposite result.
Bernard Woolley: How?
Sir Humphrey: Mr. Woolley, are you worried about the danger of war?
Bernard Woolley: Yes
Sir Humphrey: Are you worried about the growth of armaments?
Bernard Woolley: Yes
Sir Humphrey: Do you think there is a danger in giving young people guns and teaching them how to kill?
Bernard Woolley: Yes
Sir Humphrey: Do you think it is wrong to force people to take up arms against their will?
Bernard Woolley: Yes
Sir Humphrey: Would you oppose the reintroduction of National Service?
Bernard Woolley: Yes
Sir Humphrey: There you are, you see Bernard. The perfect balanced sample.

Play

Solving the Cubic $ax^3 + bx^2 + cx + d = 0$

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Δ = -18 a b c d + 4 b³ d - b² c² + 4 a c³ + 27 a² d²;
δ = √Δ;
Γ = 2 b³ - 9 a b c + 27 a² d + 3 √3 a δ;
γ = ∛(Γ / 2);
r = - (b + γ + (b² - 3 a c) / γ) / (3 a)
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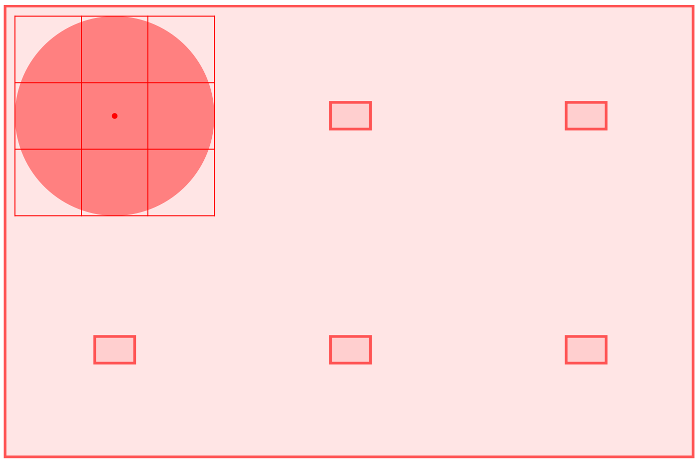
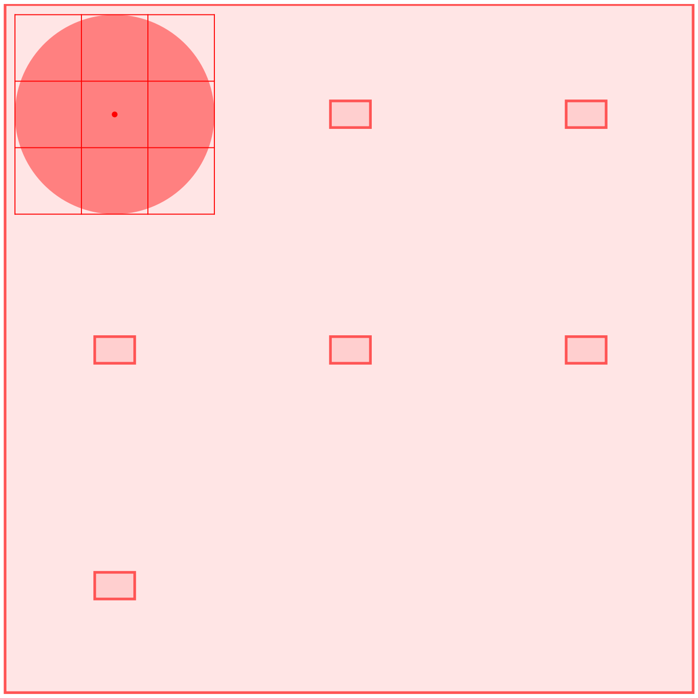


Testing the Cubic Solution

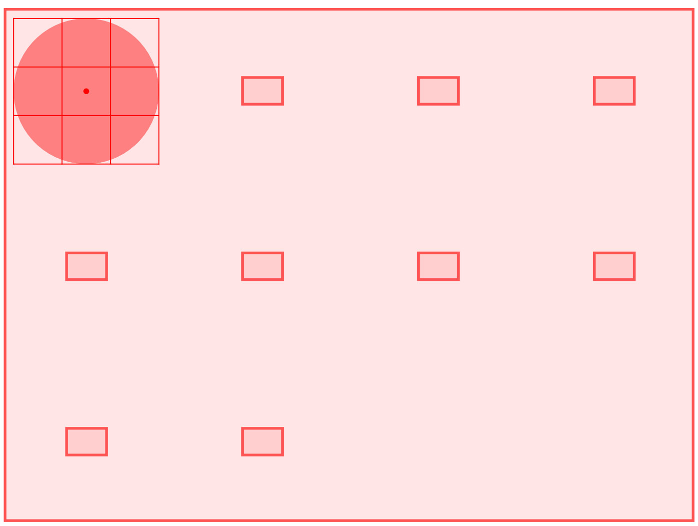


The phenomena observed, that the output r *always* follows one of the λ's, is *provable*.

Note. A swap-dance for λ_2 and λ_3 becomes a return-dance of a, b, c, d, yet a one-way dance for r.

Solving the Quartic $ax^4 + bx^3 + cx^2 + dx + e = 0$

$$\Delta_0 = c^2 - 3\,b\,d + 12\,a\,e;$$
$$\Delta_1 = 2\,c^3 - 9\,b\,c\,d + 27\,b^2\,e + 27\,a\,d^2 - 72\,a\,c\,e;$$
$$\Delta_2 = \left(-4\,\Delta_0{}^3 + \Delta_1{}^2\right) / 27;$$
$$u = \left(8\,a\,c - 3\,b^2\right) / \left(8\,a^2\right);$$
$$v = \left(b^3 - 4\,a\,b\,c + 8\,a^2\,d\right) / \left(8\,a^3\right);$$
$$\delta_2 = \sqrt{\Delta_2};$$
$$Q = \left(\Delta_1 + 3\,\sqrt{3}\,\delta_2\right) / 2;$$
$$q = \sqrt[3]{Q};$$
$$S = -u / 6 + (q + \Delta_0 / q) / (12\,a);$$
$$s = \sqrt{S};$$
$$\Gamma = -4\,S - 2\,u - v / s;$$
$$\gamma = \sqrt{\Gamma};$$
$$r = -b / (4\,a) + s + \gamma / 2$$



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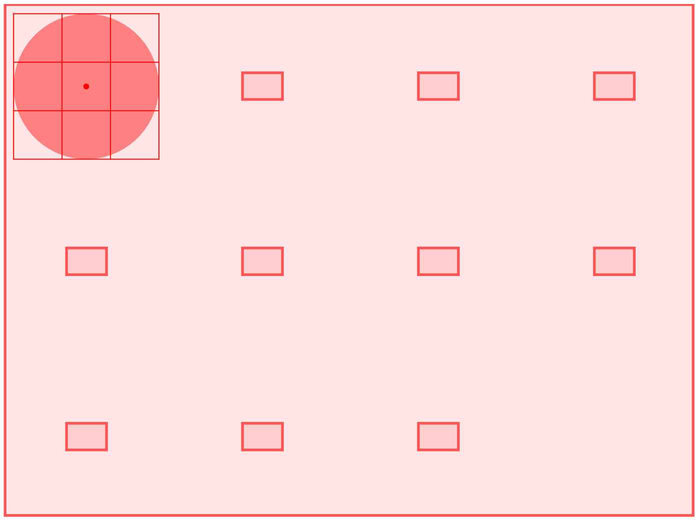
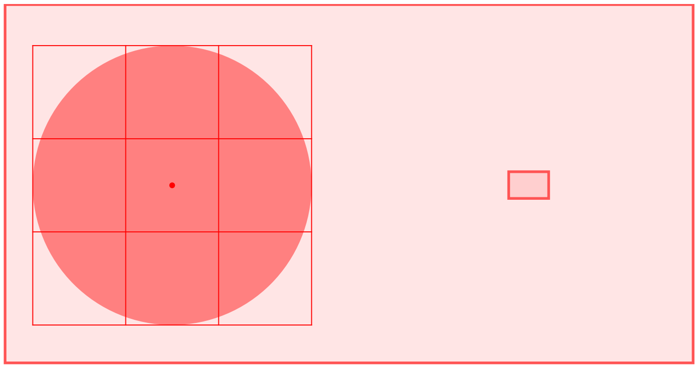


Theorem

No such machine exists for the quintic,

$$ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0.$$

The 10th Root

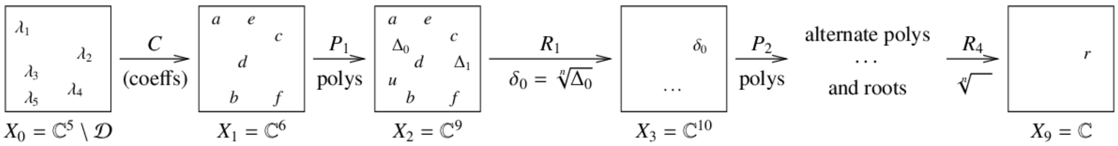


The Key Point

The persistent root of a closed path is not necessarily a closed path, yet if a closed path is the commutator of two closed paths, its persistent root is a closed path.

Proof

Proof. Suppose there was a formula, and consider the corresponding "composition of machines" picture:



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V.I. Arnold

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Advantages / Disadvantages

This proof is much simpler than the one usually presented in Galois theory classes, and in some sense it is more general - not only we show that the quintic is not soluble in radicals; in fact, the same proof also shows that the quintic is not soluble using any collection of univalent functions: exp, sin, ζ, and even log.

Yet one thing the classical proof does and we don't: Classical Galois theory can show, and we can't, that a specifc equation, say $x^5 - x + 1 = 0$, cannot be solved using the basic operations and roots.