StonyBrook-1805 handout on 180507

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Dror Bar-Natan: Talks: StonyBrook-1805: Thanks for inviting me to the SCGP!

Computation without Representation

ωεβ:=http://drorbn.net/sb18/

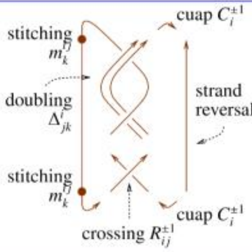
Follows Rozansky [Ro1, Ro2, Ro3] and Overbay [Ov], joint with van der Veen. More at ωεβ/talks.

Abstract. A major part of "quantum topology" is the definition and computation of various knot invariants by carrying out computations in quantum groups. Traditionally these computations are carried out "in a representation", but this is very slow: one has to use tensor powers of these representations, and the dimensions of powers grow exponentially fast. I will describe a direct-participation method for carrying out these computations without having to choose a representation and explain why in many ways the results are better and faster. The two key points we use are a technique for composing infinite-order "perturbed Gaussian" differential operators, and the little-known fact that every semi-simple Lie algebra can be approximated by solvable Lie algebras, where computations are easier.

The Knot Theory Portfolio.

- Has operations $\sqcup$, m, Δ, S.
- All tangles are generated by $R^{\pm 1}$ and $C^{\pm}$ (so "easy" to produce invariants).
- Makes some properties ("genus", "ribbon") be "definable".

(more to say, but not now).



A "Quantum Group" Portfolio consists of an algebra U along with maps (and some axioms...)

$\sqcup \to \otimes$

$$\begin{tikzcd}
\mathbb{Q} = \hat{U}^{\otimes\emptyset} \arrow[r, "C_i"] & \hat{U}^{\otimes\{i\}} \arrow[loop above, "S_i"] \arrow[r, bend left, "\Delta^i_{jk}"] & \hat{U}^{\otimes\{j,k\}} \arrow[l, bend left, "m_i^{jk}"] & \mathbb{Q} = \hat{U}^{\otimes\emptyset} \arrow[l, "R_{jk}"'] \\
\hat{\mathcal{S}}(\emptyset) \arrow[r, "C_i"] \arrow[u, "\mathbb{O}_0"] & \hat{\mathcal{S}}(B_i) \arrow[loop below, "S_i"] \arrow[u, "\mathbb{O}_{y_ix_i\cdots}"] \arrow[r, bend left, "\Delta^i_{jk}"] & \hat{\mathcal{S}}\left(B_j, B_k\right) \arrow[u, "\mathbb{O}_{y_jx_j\cdots\otimes y_kx_k\cdots}"] \arrow[l, bend left, "m_i^{jk}"] & \hat{\mathcal{S}}(\emptyset) \arrow[l, "R_{jk}"'] \arrow[u, "\mathbb{O}_0"]
\end{tikzcd}$$

$\sqcup \to \cdot$

(of "generators")

PBW Bases. The U's we care about always have "Poincaré-Birkhoff-Witt" bases; there is some finite set B and isomorphisms $\mathbb{O}_{y,x,\dots}\colon \hat{\mathcal{S}}(B) \to U$ defined by "ordering monomials" to some fixed $y, x, \dots$ order. The quantum group portfolio now becomes a "symmetric algebra" potfolio, or a "power series" portfolio.

The Category $\mathcal{DO}$. Hence we care about the monoidal category $\mathcal{DO}$ whose objects are finite sets B and whose morphisms are $\mathrm{mor}_{\mathcal{DO}}(B, B') := \mathrm{Hom}_{\mathbb{Q}}(\mathcal{S}(B) \to \mathcal{S}(B')) = \mathcal{S}(B^*, B')$ (by convention, $x^* = \xi$, $y^* = \eta$, etc.).

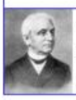
The Composition Law. If

$$\mathcal{S}(B_0) \xrightarrow[{}^t f \in \mathbb{Q}[\![\zeta_{0i}, z_{1j}]\!]]{f} \mathcal{S}(B_1) \xrightarrow[{}^t g \in \mathbb{Q}[\![\zeta_{1j}, z_{2k}]\!]]{g} \mathcal{S}(B_2)$$

A

then ${}^t(f /\!\!/ g) = {}^t(g \circ f) = \left(g|_{\zeta_{1j} \to \partial_{z_{1j}}} f \right)_{z_{1j}=0} = \left(f|_{z_{1j} \to \partial_{\zeta_{1j}}} g \right)_{\zeta_{1j}=0}$.

Proposition. If $F\colon \mathcal{S}(B) \to \mathcal{S}(B')$ is linear and "continuous", then ${}^t F = \exp\left(\sum_{z_i \in B} \zeta_i z_i\right) /\!\!/ F$.



"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

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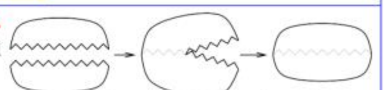
Examples.

1. The 1-variable identity map $I\colon \mathcal{S}(z) \to \mathcal{S}(z)$ is given by ${}^t I_1 = \mathbb{e}^{z\zeta}$ and the n-variable one by ${}^t I_n = \mathbb{e}^{z_1\zeta_1 + \cdots + z_n\zeta_n}$.
2. The "$z_i \to z_j$ variable rename map $\sigma^i_j\colon \mathcal{S}(z_i) \to \mathcal{S}(z_j)$" becomes ${}^t\sigma^i_j = \mathbb{e}^{z_j\zeta_i}$, and it's easy to rename several variables simultaneously.
3. The "archetypal multiplication map $m_k^{ij}\colon \mathcal{S}(z_i, z_j) \to \mathcal{S}(z_k)$" has ${}^t m = \mathbb{e}^{z_k(\zeta_i + \zeta_j)}$.
4. The "archetypal coproduct $\Delta^i_{jk}\colon \mathcal{S}(z_i) \to \mathcal{S}(z_j, z_k)$", given by $z_i \to z_j + z_k$ or $\Delta z = z \otimes 1 + 1 \otimes z$, has ${}^t\Delta = \mathbb{e}^{(z_j+z_k)\zeta_i}$.
5. R-matrices tend to have terms of the form $\mathbb{e}_q^{\hbar y_1 x_2} \in \mathcal{U}_q \otimes \mathcal{U}_q$. The "baby R-matrix" is ${}^t R = \mathbb{e}^{\hbar y x} \in \mathcal{S}(y, x)$.
6. The "Weyl form of the canonical commutation relations" states that if $[y, x] = t$ is a scalar, then $\mathbb{e}^{\xi x}\mathbb{e}^{\eta y} = \mathbb{e}^{\eta y}\mathbb{e}^{\xi x}\mathbb{e}^{-\eta\xi t}$. Thus with

$$\begin{tikzcd}
\mathcal{S}(y, x) \arrow[loop left, "SW_{xy}"] \arrow[r, shift left, "\mathbb{O}_{xy}"] \arrow[r, shift right, "\mathbb{O}_{yx}"'] & \mathcal{U}(y, x)
\end{tikzcd}$$

we have ${}^t SW_{xy} = \mathbb{e}^{\eta y + \xi x - \eta\xi t}$.

The Zipping Issue. (between unbound and bound lies half-zipped).



Zipping. If $P(\zeta^j, z_i)$ is a polynomial, or whenever otherwise convergent, set

$$\left\langle P(\zeta^j, z_i) \right\rangle_{(\zeta^j)} = P\left(\partial_{z_j}, z_i\right)\Big|_{z_i=0}.$$

(E.g., if $P = \sum a_{nm}\zeta^n z^m$ then $\langle P \rangle_\zeta = \sum n! a_{nn}$).

Implementation. ωεβ/Zip

```
z* = ζ; ζ* = z; Zip{}[P_] := P;
Zip{ζ_,ζs___}[P_] :=
 (Expand[P // Zip{ζs}] /. f_. ζ^d_. ⧴ ∂{ζ*,d} f) /. ζ* → 0
{Zip{ζ}[ζ^2 e^(δ z^2)], Zip{ζ}[ζ^4 e^(δ z^2)]}
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{2 δ, 12 δ^2}

Note: For $f \in S(z)$ $g \in S(\zeta)$,

A: $\langle f(z), g(\zeta) \rangle = f(\partial_\zeta) g|_{\zeta=0} = g(\partial_z) f|_{z=0}$

The Zipping / Contraction Theorem. If P has a finite ζ-degree and the y's and the q's are "small" then

$$\left\langle P(z_i,\zeta^j)\mathbb{e}^{\eta^j z_i + y_j\zeta^j}\right\rangle_{(\zeta^j)} = \left\langle P(z_i+y_i,\zeta^j)\mathbb{e}^{\eta^j(z_i+y_i)}\right\rangle_{(\zeta^j)},$$

(proof: replace $y_j \to \hbar y_j$ and test at $\hbar=0$ and at $\partial_\hbar$), and

$$\left\langle P(z_i,\zeta^j)\mathbb{e}^{c+\eta^j z_i + y_j\zeta^j + q^i_j z_i\zeta^j}\right\rangle_{(\zeta^j)} = \det(\tilde q)\left\langle P(\tilde q^k_i(z_k+y_k),\zeta^j)\mathbb{e}^{c+\eta^j\tilde q^k_i(z_k+y_k)}\right\rangle_{(\zeta^j)}$$

where $\tilde q$ is the inverse matrix of $1-q$: $(\delta^i_j - q^i_j)\tilde q^j_k = \delta^i_k$ (proof: replace $q^i_j \to \hbar q^i_j$ and test at $\hbar=0$ and at $\partial_\hbar$).

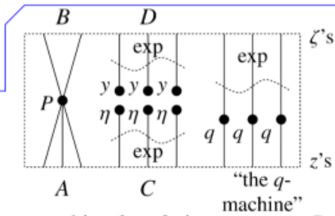
make space

Exponential Reservoirs.. The true Hilbert hotel is exp! Remove one x from an "exponential reservoir" of x's and you are left with the same exponential reservoir:

$$\mathbb{e}^x = \left[\ldots + \frac{xxxxx}{120} + \ldots\right] \xrightarrow{\partial_x} \left[\ldots + \frac{\cancel{x}xxxx}{120} + \ldots\right] = (\mathbb{e}^x)' = \mathbb{e}^x,$$

and if you let each element choose left or right, you get twice the same reservoir:

$$\mathbb{e}^x \xrightarrow{x\to x_l+x_r} \mathbb{e}^{x_l+x_r} = \mathbb{e}^{x_l}\mathbb{e}^{x_r}.$$



A Graphical Proof. Glue top to bottom on the right, in all possible ways. Several scenarios occur:

1. Start at A, go through the q-machine $k \geq 0$ times, stop at B. Get $\left\langle P\left(\sum_{k\geq 0} q^k z,\zeta\right)\right\rangle = \langle P(\tilde q z,\zeta)\rangle$.
2. Loop through the q-machine and swallow your own tail. Get $\exp\left(\sum q^k/k\right) = \exp(-\log(1-q)) = \tilde q$.
3. …

By the reservoir splitting principle, these scenarios contribute multiplicatively. □

Implementation. ωεβ/Zip

```
𝔼 /: Zip_{ζs_List}@𝔼[Q_, P_] :=      (* 𝔼[Q,P] means e^Q P *)
  Module[{ζ, z, zs, c, ys, ηs, qt, zrule, Q1, Q2},
   zs = Table[ζ*, {ζ, ζs}];
   c = Q /. Alternatives @@ (ζs ⋃ zs) → 0;
   ys = Table[∂_ζ (Q /. Alternatives @@ zs → 0), {ζ, ζs}];
   ηs = Table[∂_z (Q /. Alternatives @@ ζs → 0), {z, zs}];
   qt = Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ}Q, {ζ, ζs}, {z, zs}];
   zrule = Thread[zs → qt.(zs + ys)];
   Q1 = c + ηs.zs /. zrule; Q2 = Q1 /. Alternatives @@ zs → 0;
   Simplify /@ 𝔼[Q2, Det[qt] e^{-Q2} Zip_{ζs}[e^{Q1} (P /. zrule)]] ];
```

The Real Thing. In the algebra QU_ϵ (explained later), over $\mathbb{Q}[[\hbar]]$ using the *yaxt* order, $T=\mathbb{e}^{\hbar t}$, $\bar T = T^{-1}$, $\mathcal{A}=\mathbb{e}^{\alpha}$, and $\bar{\mathcal{A}} = \mathcal{A}^{-1}$, we have

$${}^tR_{ij} = \mathbb{e}^{\hbar(y_ix_j - t_ia_j)}\left(1+\epsilon\hbar\left(a_ia_j - \hbar^2y_i^2x_j^2/4\right) + O(\epsilon^2)\right)$$

in $\mathcal{S}(B_i,B_j)$, and in $\mathcal{S}(B_1^*,B_2^*,B)$ we have

$${}^tm = \mathbb{e}^{(\alpha_1+\alpha_2)a + \eta_2\xi_1(1-T)/\hbar + (\xi_1\bar{\mathcal{A}}_2+\xi_2)x + (\eta_1+\eta_2\bar{\mathcal{A}}_1)y}\left(1+\epsilon\lambda+O(\epsilon^2)\right),$$

where $\lambda = 2a\eta_2\xi_1T + \eta_2^2\xi_1^2\left(3T^2-4T+1\right)/4\hbar - \eta_2\xi_1^2(3T-1)x\bar{\mathcal{A}}_2/2 - \eta_2^2\xi_1(3T-1)y\bar{\mathcal{A}}_1/2 + \eta_2\xi_1xy\hbar\bar{\mathcal{A}}_1\bar{\mathcal{A}}_2$,

$${}^t\Delta = \mathbb{e}^{\tau(t_1+t_1)+\eta(y_1+T_1y_2)+\alpha(a_1+a_2)+\xi(x_1+x_2)}(1+O(\epsilon)) \in \mathcal{S}(B^*,B_1,B_2),$$

and

$${}^tS = \mathbb{e}^{-\tau t - \alpha a - \eta\xi(1-\bar T)\mathcal{A}/\hbar - \bar T\eta y\mathcal{A} - \xi x\mathcal{A}}(1+O(\epsilon)) \in \mathcal{S}(B^*,B).$$

Real Zipping is a minor mess, and is done in two phases:

	τa-phase		ξy-phase	
ζ-like variables	τ	a	ξ	y
z-like variables	t	α	x	η

Already at $\epsilon=0$ we get the best known formulas for the Alexander polynomial! Docility

Generic ~~Definition.~~ A "docile perturbed Gaussian" in the variables $(z_i)_{i\in S}$ over the ring R is an expression of the form

$$\mathbb{e}^{q^{ij}z_iz_j}P = \mathbb{e}^{q^{ij}z_iz_j}\left(\sum_{k\geq 0}\epsilon^kP_k\right),$$

where all coefficients are in R and where P is a "docile series": $\deg P_k \leq 4k$.

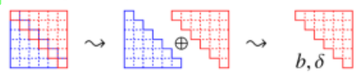
Our Docility. In the case of QU_ϵ, all invariants and operations are of the form $\mathbb{e}^{L+Q}P$, where

- L is a quadratic of the form $\sum l_{z\zeta}z\zeta$, where z runs over $\{t_i,\alpha_i\}_{i\in S}$ and ζ over $\{\tau_i,a_i\}_{i\in S}$, with integer coefficients $l_{z\zeta}$.
- Q is a quadratic for the form $\sum q_{z\zeta}z\zeta$, where z runs over $\{x_i,\eta_i\}_{i\in S}$ and ζ over $\{\xi_i,y_i\}_{i\in S}$, with coefficients $q_{z\zeta}$ in the ring R_S of rational functions in $\{T_i,\mathcal{A}_i\}_{i\in S}$.
- P is a docile power series in $\{y_i,a_i,x_i,\eta_i,\xi_i\}_{i\in S}$ with coefficients in R_S, and where $\deg(y_i,a_i,x_i,\eta_i,\xi_i) = (1,2,1,1,1)$.

Docililty Matters! The rank of the space of docile series to ϵ^k is polynomial in the number of variables $|S|$. DDD!!

- At $\epsilon^2=0$ we get the Rozansky-Overbay [Ro1, Ro2, Ro3, Ov] invariant, which is stronger than HOMFLY-PT polynomial and Khovanov homology taken together!
- In general, get "higher diagonals in the Melvin-Morton-Rozansky expansion of the coloured Jones polynomial" [MM, BNG], but why spoil something good?

Solvable Approximation. In gl_n, half is enough! Indeed $gl_n \oplus \mathfrak{a}_n = \mathcal{D}(◹, b, \delta)$:



$b(◹) = b$: ◹ ⊗ ◹ → ◹
$b(◺) \rightsquigarrow \delta$: ◹ → ◹ ⊗ ◹

Now define $gl_n^\epsilon := \mathcal{D}(◹, b, \epsilon\delta)$. Schematically, this is $[◹,◹]=◹$, $[◺,◺]=\epsilon◺$, and $[◹,◺] = ◺ + \epsilon◹$. The same process works for all semi-simple Lie algebras, and at $\epsilon^{k+1}=0$ always yields a solvable Lie algebra.

***CU* and *QU*.** Starting from sl_2, get $CU_\epsilon = \langle y,a,x,t\rangle/([t,-]=0,\ [a,y]=-y,\ [a,x]=x,\ [x,y]=2\epsilon a - t)$. Quantize using standard tools (I'm sorry) and get $QU_\epsilon = \langle y,a,x,t\rangle/([t,-]=0,\ [a,y]=-y,\ [a,x]=x,\ xy - \mathbb{e}^{\hbar\epsilon}yx = (1-T\mathbb{e}^{-2\hbar\epsilon a})/\hbar)$.

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