StonyBrook-1805 handout on 180506

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Dror Bar-Natan: Talks: StonyBrook-1805:

Computation without Representation

Thanks for inviting me to the SCGP! $\omega\varepsilon\beta$:=http://drorbn.net/sb18/

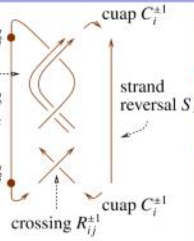
Follows Rozansky [Ro1, Ro2, Ro3] and Overbay [Ov], joint with van der Veen. More at $\omega\varepsilon\beta$/talks.

Abstract. A major part of "quantum topology" is the definition and computation of various knot invariants by carrying out computations in quantum groups. Traditionally these computations are carried out "in a representation", but this is very slow: one has to use tensor powers of these representations, and the dimensions of powers grow exponentially fast. I will describe a direct-participation method for carrying out these computations without having to choose a representation and explain why in many ways the results are better and faster. The two key points we use are a technique for composing infinite-order "perturbed Gaussian" differential operators, and the little-known fact that every semi-simple Lie algebra can be approximated by solvable Lie algebras, where computations are easier.

The Knot Theory Portfolio.

- Has operations $\sqcup, m, \Delta, S$.
- All tangles are generated by $R^{\pm1}$ and $C^{\pm}$ (so "easy" to produce invariants).
- Makes some properties ("genus", "ribbon") be "definable".

(more to say, but not now).



A "Quantum Group" Portfolio consists of an algebra U along with maps (and some axioms...)

$\sqcup \to \otimes$

$$\begin{tikzcd}
\mathbb{Q}=\hat{U}^{\otimes\emptyset} \arrow[r, "C_i"] & \hat{U}^{\otimes\{i\}} \arrow[loop above, "S_i"] \arrow[r, bend left, "\Delta^i_{jk}"] & \hat{U}^{\otimes\{j,k\}} \arrow[l, bend left, "m^{jk}_i"] & \mathbb{Q}=\hat{U}^{\otimes\emptyset} \arrow[l, "R_{jk}"'] \\
\hat{\mathcal{S}}(\emptyset) \arrow[u, "\mathbb{O}_{()}"] \arrow[r, "C_i"] & \hat{\mathcal{S}}(B_i) \arrow[u, "\mathbb{O}_{y_ix_i\cdots}"] \arrow[loop below, "S_i"] \arrow[r, bend left, "\Delta^i_{jk}"] & \hat{\mathcal{S}}(B_j,B_k) \arrow[u, "\mathbb{O}_{y_jx_j\cdots\otimes y_kx_k\cdots}"] \arrow[l, bend left, "m^{jk}_i"] & \hat{\mathcal{S}}(\emptyset) \arrow[u, "\mathbb{O}_{()}"] \arrow[l, "R_{jk}"']
\end{tikzcd}$$

$\sqcup \to \cdot$

PBW Bases. The U's we care about always have "Poincaré-Birkhoff-Witt" bases; there is some finite set B and isomorphisms $\mathbb{O}_{y,x,\ldots}\colon \hat{\mathcal{S}}(B)\to U$ defined by "ordering monomials" to some fixed $y,x,\ldots$ order. The quantum group portfolio now becomes a "symmetric algebra" potfolio, or a "power series" portfolio.

The Category $\mathcal{DO}$. Hence we care about the monoidal category $\mathcal{DO}$ whose objects are finite sets B and whose morphisms are $\mathrm{mor}_{\mathcal{DO}}(B,B') := \mathrm{Hom}_{\mathbb{Q}}(\mathcal{S}(B)\to\mathcal{S}(B')) = \mathcal{S}(B^*,B')$ (by convention, $x^*=\xi$, $y^*=\eta$, etc.).

The Composition Law. If

$$\mathcal{S}(B_0) \xrightarrow[{}^t f\in\mathbb{Q}[\![\zeta_{0i},z_{1j}]\!]]{f} \mathcal{S}(B_1) \xrightarrow[{}^t g\in\mathbb{Q}[\![\zeta_{1j},z_{2k}]\!]]{g} \mathcal{S}(B_2)$$

then ${}^t(f/\!\!/g) = {}^t(g\circ f) = \left(g|_{\zeta_{1j}\to\partial_{z_{1j}}} f\right)_{z_{1j}=0} = \left(f|_{z_{1j}\to\partial_{\zeta_{1j}}} g\right)_{\zeta_{1j}=0}$.

Proposition. If $F\colon\mathcal{S}(B)\to\mathcal{S}(B')$ is linear and "continuous", then ${}^tF = \exp\left(\sum_{z_i\in B}\zeta_i z_i\right)/\!\!/F$.

"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

www.katlas.org 

Examples.

1. The 1-variable identity map $I\colon\mathcal{S}(z)\to\mathcal{S}(z)$ is given by ${}^tI_1 = \mathbb{e}^{z\zeta}$ and the n-variable one by ${}^tI_n = \mathbb{e}^{z_1\zeta_1+\cdots+z_n\zeta_n}$.
2. The "$z_i\to z_j$ variable rename map $\sigma^i_j\colon\mathcal{S}(z_i)\to\mathcal{S}(z_j)$ becomes ${}^t\sigma^i_j = \mathbb{e}^{z_j\zeta_i}$, and it's easy to rename several variables simultaneously.
3. The "archetypal multiplication map $m^{ij}_k\colon\mathcal{S}(z_i,z_j)\to\mathcal{S}(z_k)$" has ${}^tm = \mathbb{e}^{z_k(\zeta_i+\zeta_j)}$.
4. The "archetypal coproduct $\Delta^i_{jk}\colon\mathcal{S}(z_i)\to\mathcal{S}(z_j,z_k)$", given by $z_i\to z_j+z_k$ or $\Delta z = z\otimes1+1\otimes z$, has ${}^t\Delta = \mathbb{e}^{(z_j+z_k)\zeta_i}$.
5. R-matrices tend to have terms of the form $\mathbb{e}_q^{\hbar y_1x_2}\in\mathcal{U}_q\otimes\mathcal{U}_q$. The "baby R-matrix" is ${}^tR = \mathbb{e}^{\hbar yx}\in\mathcal{S}(y,x)$.
6. The "Weyl form of the canonical commutation relations" states that if $[y,x]=t$ is a scalar, then $\mathbb{e}^{\xi x}\mathbb{e}^{\eta y} = \mathbb{e}^{\eta y}\mathbb{e}^{\xi x}\mathbb{e}^{-\eta\xi t}$. Thus with

$$\begin{tikzcd}
\mathcal{S}(y,x) \arrow[loop left, "SW_{xy}"] \arrow[r, shift left, "\mathbb{O}_{xy}"] \arrow[r, shift right, "\mathbb{O}_{yx}"'] & \mathcal{U}(y,x)
\end{tikzcd}$$

we have ${}^tSW_{xy} = \mathbb{e}^{\eta y+\xi x-\eta\xi t}$.

The Zipping Issue. (between unbound and bound lies half-zipped).

Zipping. If $P(\zeta^j,z_i)$ is a polynomial, or whenever otherwise convergent, set

$$\left\langle P(\zeta^j,z_i)\right\rangle_{(\zeta^j)} = P\left(\partial_{z_j},z_i\right)\Big|_{z_i=0}.$$

(E.g., if $P=\sum a_{nm}\zeta^nz^m$ then $\langle P\rangle_\zeta = \sum n!a_{nn}$).

Implementation. $\omega\varepsilon\beta$/Zip

```
z* = ζ; ζ* = z; Zip{}[P_] := P;
Zip{ζ_,ζs___}[P_] :=
  (Expand[P // Zip{ζs}] /. f_. ζ^d_. :→ ∂{ζ*,d} f) /. ζ* → 0
{Zip{ζ}[ζ^2 e^(δ z^2)], Zip{ζ}[ζ^4 e^(δ z^2)]}
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$\{2\delta, 12\delta^2\}$

The Zipping / Contraction Theorem. If P has a finite ζ-degree and the y's and the q's are "small" then

$$\left\langle P(z_i,\zeta^j)\mathbb{e}^{\eta^iz_i+y_j\zeta^j}\right\rangle_{(\zeta^j)} = \left\langle P(z_i+y_i,\zeta^j)\mathbb{e}^{\eta^i(z_i+y_i)}\right\rangle_{(\zeta^j)},$$

(proof: replace $y_j\to\hbar y_j$ and test at $\hbar=0$ and at $\partial_\hbar$), and

$$\left\langle P(z_i,\zeta^j)\mathbb{e}^{c+\eta^iz_i+y_j\zeta^j+q^i_jz_i\zeta^j}\right\rangle_{(\zeta^j)}$$

Zipping is a minor mess!

phase 1 phase 2

ζ	t a	ξ y

(proof: replace $y_j \to \hbar y_j$ and test at $\hbar = 0$ and at $\partial_\hbar$), and

$$\left\langle P(z_i, \zeta^j) \mathbb{e}^{c+\eta^i z_i + y_j \zeta^j + q_j^i z_i \zeta^j} \right\rangle_{(\zeta^j)} = \det(\tilde{q}) \left\langle P(\tilde{q}_i^k (z_k + y_k), \zeta^j) \mathbb{e}^{c+\eta^i \tilde{q}_i^k (z_k+y_k)} \right\rangle_{(\zeta^j)}$$

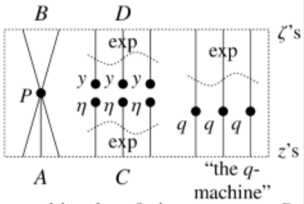
where $\tilde{q}$ is the inverse matrix of $1 - q$: $(\delta_j^i - q_j^i)\tilde{q}_k^j = \delta_k^i$ (proof: replace $q_j^i \to \hbar q_j^i$ and test at $\hbar = 0$ and at $\partial_\hbar$).

Exponential Reservoirs.. The true Hilbert hotel is exp! Remove one x from an "exponential reservoir" of x's and you are left with the same exponential reservoir:

$$\mathbb{e}^x = \left[\ldots + \frac{xxxxx}{120} + \ldots\right] \xrightarrow{\partial_x} \left[\ldots + \frac{x\cancel{x}xxx}{120} + \ldots\right] = (\mathbb{e}^x)' = \mathbb{e}^x,$$

and if you let each element choose left or right, you get twice the same reservoir:

$$\mathbb{e}^x \xrightarrow{x \to x_l + x_r} \mathbb{e}^{x_l + x_r} = \mathbb{e}^{x_l}\mathbb{e}^{x_r}.$$



A Graphical Proof. Glue top to bottom on the right, in all possible ways. Several scenarios occur:

1. Start at A, go through the q-machine $k \geq 0$ times, stop at B. Get $\left\langle P\left(\sum_{k\geq 0} q^k z, \zeta\right)\right\rangle = \langle P(\tilde{q}z, \zeta)\rangle$.
2. Loop through the q-machine and swallow your own tail. Get $\exp\left(\sum q^k/k\right) = \exp(-\log(1-q)) = \tilde{q}$.
3. ...

By the reservoir splitting principle, these scenarios contribute multiplicatively. □

one block implementation.

the algebra explained later,

The Real Thing. In $QU_\epsilon(\epsilon^2 = 0)$ over $\mathbb{Q}[\![\hbar]\!]$ using the yax order, $T = \mathbb{e}^{\hbar t}$, $\bar{T} = T^{-1}$, $\mathcal{A} = \mathbb{e}^{\gamma\alpha}$, and $\bar{\mathcal{A}} = \mathcal{A}^{-1}$, we have

$${}^t R_{ij} = \mathbb{e}^{\hbar(y_i x_j - t_i a_j/\gamma)} \left(1 + \epsilon\hbar\left(a_i a_j/\gamma - \gamma\hbar^2 y_i^2 x_j^2/4\right)\right) + O(\epsilon^2)$$

de-γ!

in $\mathcal{S}(B_i, B_j)$, and in $\mathcal{S}(B_1^*, B_2^*, B)$ we have

$${}^t m = \mathbb{e}^{(\alpha_1+\alpha_2)a + \eta_2\xi_1(1-T)/\hbar + (\xi_1\bar{\mathcal{A}}_2 + \xi_2)x + (\eta_1 + \eta_2\bar{\mathcal{A}}_1)y} (1 + \epsilon\lambda_m) + O(\epsilon^2),$$

where $\lambda_m = 2a\eta_2\xi_1 T + \frac{1}{4}\gamma\eta_2^2\xi_1^2(3T^2 - 4T + 1)/\hbar - \frac{1}{2}\gamma\eta_2\xi_1^2(3T-1)x\bar{\mathcal{A}}_2 - \frac{1}{2}\gamma\eta_2^2\xi_1(3T-1)y\bar{\mathcal{A}}_1 + \gamma\eta_2\xi_1 xy\hbar\bar{\mathcal{A}}_1\bar{\mathcal{A}}_2$. ~~Similar formulas delight us for ${}^t\Delta$ and ${}^t S$.~~

$\Delta^t = (\quad) + O(\epsilon)$
${}^t S = (\quad) + O(\epsilon)$
From sl_2 Portfolio

Sketch. (Total 57m)

1. (5m) The knot theory portfolio.
2. (5m) The (quantum) group portfolio. (Write in general-algebra language, not in universal enveloping algebra language).
3. (3m) Quantum groups have "PBW" basis. Hence they are symmetric algebras, with funny products / co-products.
4. (3m) The DO category.
5. (5m) Example: the Abelian case.
6. (3m) Example: k=0.
7. (5m) Example: sl2.
8. (10m) Zipping and composing. The zipping theorem.
9. (5m) Docility and polynomiallity.
10. (5m) Solvable Approximation.
11. (8m) Mention Rozansky-Overbay, mention computations.

ζ	T a	ξ y
z	t α	x η

At $\epsilon = 0$ this becomes the best known formulas for the Alexander polynomial!
At $\epsilon^2 = 0$ it is stronger than HOFFLY-PT & Khanov taken together!

The Rozansky overbay invariant,

Docility and Polynomiality.
From Matemale/Lecture 3 handout.
As below

Solvable approximations as in Matemale & as below.

References.

[BV1] D. Bar-Natan and R. van der Veen, *A Polynomial Time Knot Polynomial*, arXiv:1708.04853.

[Ov] A. Overbay, *Perturbative Expansion of the Colored Jones Polynomial*, University of North Carolina PhD thesis, ωεβ/Ov.

[Ro1] L. Rozansky, *A contribution of the trivial flat connection to the Jones polynomial and Witten's invariant of 3d manifolds, I*, Comm. Math. Phys. **175-2** (1996) 275–296, arXiv:hep-th/9401061.

[Ro2] L. Rozansky, *The Universal R-Matrix, Burau Representation and the Melvin-Morton Expansion of the Colored Jones Polynomial*, Adv. Math. **134-1** (1998) 1–31, arXiv:q-alg/9604005.

[Ro3] L. Rozansky, *A Universal U(1)-RCC Invariant of Links and Rationality Conjecture*, arXiv:math/0201139.

Generic

Definition. A "docile perturbed Gaussian" in the variables $(z_i)_{i\in S}$ over the ring R is an expression of the form

doc·ile
/ˈdäsəl/
adjective
ready to accept control or instruction; submissive
"a cheap and docile workforce"

$$\mathbb{e}^{q^{ij}z_i z_j} P = \mathbb{e}^{q^{ij}z_i z_j}\left(\sum_{k\geq 0} \epsilon^k P_k\right),$$

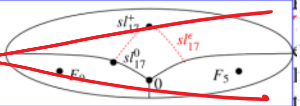
where all coefficients are in R and where P is a "docile series": $\deg P_k \leq 4k$.

Docililty Matters! The rank of the space of docile series to ϵ^k is polynomial in the number of variables $|S|$.

In ~~our case~~ the QUε our invariants and operations are of the form $\mathbb{e}^{L+Q}P$, where

- L is a quadratic of the form $\sum l_{z\zeta} z\zeta$, where z runs over $\{t_i, \alpha_i\}_{i\in S}$ and ζ runs over $\{\tau_i, a_i\}_{i\in S}$, with integer coefficients $l_{z\zeta}$.
- Q is a quadratic of the form $\sum q_{z\zeta} z\zeta$, where z runs over $\{x_i, \eta_i\}_{i\in S}$ and ζ runs over $\{\xi_i, y_i\}_{i\in S}$, with coefficients $q_{z\zeta}$ in the ring R_s of rational functions in $(T_i)_{i\in S}$ and $(\mathcal{A}_i)_{i\in S}$.
- $P = \sum \epsilon^k P_k$ is a docile power series in $\{y_i, a_i, x_i, \eta_i, \xi_i\}_{i\in S}$, where $\deg(y_i, a_i, x_i, \eta_i, \xi_i) = (1, 2, 1, 1, 1)$, with coefficients in R_s.

~~**The (fake) moduli** of Lie algebras on V, a quadratic variety in $(V^*)^{\otimes 2}\otimes V$ is on the right. We care about $sl^k_{17} := sl^\epsilon_{17}/(\epsilon^{k+1}=0)$.~~



3

Solvable Approximation. ~~A quantized universal enveloping algebra (aka "quantum group") is an ∞-dimensional inverse limit.~~

untrimmed $\longrightarrow$ halfway-trimmed $\longrightarrow$ almost fully-trimmed

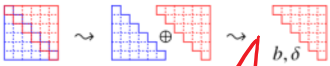
$\updownarrow$ too hard — $\updownarrow$ "solvable approximation" ~~(a parameter is hidden)~~ — $\updownarrow$ "finite-type"

reduce mod

keep h reduce mod ϵ^{k+1}

$h^{m+1}=0$

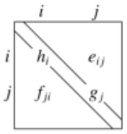
Recomposing gl_n. Half is enough! $gl_n \oplus \mathfrak{a}_n = \mathcal{D}(◹, b, \delta)$:



$b(◹) = b:\ ◹\otimes◹ \to ◹$
$b(◺) \rightsquigarrow \delta:\ ◹ \to ◹\otimes◹$

1

Now define $gl^\epsilon_n := \mathcal{D}(◹, b, \epsilon\delta)$. Schematically, this is $[◹, ◹] = ◹$, $[◺, ◺] = \epsilon◺$, and $[◹, ◺] = ◺ + \epsilon◹$. In detail, it is



$$[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{li}e_{kj} \qquad [f_{ij}, f_{kl}] = \epsilon\delta_{jk}f_{il} - \epsilon\delta_{li}f_{kj}$$
$$[e_{ij}, f_{kl}] = \delta_{jk}(\epsilon\delta_{i<l}e_{il} + \delta_{il}(h_i + \epsilon g_i)/2 + \delta_{i>l}f_{il})$$
$$-\delta_{li}(\epsilon\delta_{k<j}e_{kj} + \delta_{kj}(h_j + \epsilon g_j)/2 + \delta_{k>j}f_{kj})$$
$$[g_i, e_{jk}] = (\delta_{ij} - \delta_{ik})e_{jk} \qquad [h_i, e_{jk}] = \epsilon(\delta_{ij} - \delta_{ik})e_{jk}$$
$$[g_i, f_{jk}] = (\delta_{ij} - \delta_{ik})f_{jk} \qquad [h_i, f_{jk}] = \epsilon(\delta_{ij} - \delta_{ik})f_{jk}$$

2: This process makes sense for arbitrary semi-simple Lie algebras!

starting from sl_2, get

$CU_\epsilon = \ldots$

"quantize" with standard' tools & get

$QU_\epsilon = \ldots$

'Intimidate to submission!