StonyBrook-1805 handout on 180505

May 6, 2018 7:20 AM

Thanks for inviting me to the SCGP!

Computation without Representation

ωεβ:=http://drorbn.net/sb18/

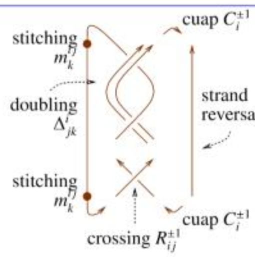
Follows Rozansky [Ro1, Ro2, Ro3] and Overbay [Ov], joint with van der Veen. More at ωεβ/talks.

Abstract. A major part of "quantum topology" is the definition and computation of various knot invariants by carrying out computations in quantum groups. Traditionally these computations are carried out "in a representation", but this is very slow: one has to use tensor powers of these representations, and the dimensions of powers grow exponentially fast. I will describe a direct-participation method for carrying out these computations without having to choose a representation and explain why in many ways the results are better and faster. The two key points we use are a technique for composing infinite-order "perturbed Gaussian" differential operators, and the little-known fact that every semi-simple Lie algebra can be approximated by solvable Lie algebras, where computations are easier.

The Knot Theory Portfolio.

- Has operations $\sqcup$, m, Δ, S.
- All tangles are generated by $R^{\pm 1}$ and $C^{\pm}$ (so "easy" to produce invariants).
- Makes some properties ("genus", "ribbon") be "definable".

(more to say, but not now).



A "Quantum Group" Portfolio consists of an algebra U along with maps (and some axioms...)

$$\begin{array}{ccccccc}
\sqcup\to\otimes & & \circlearrowleft S_i & & & & \\
\mathbb{Q}=\hat U^{\otimes\emptyset} & \xrightarrow{C_i} & \hat U^{\otimes\{i\}} & \underset{m_i^{jk}}{\overset{\Delta^i_{jk}}{\rightleftarrows}} & \hat U^{\otimes\{j,k\}} & \xleftarrow{R_{jk}} & \mathbb{Q}=\hat U^{\otimes\emptyset} \\
\uparrow{\scriptstyle \mathbb{O}_0} & & \uparrow{\scriptstyle \mathbb{O}_{y_ix_i\cdots}} & & \uparrow{\scriptstyle \mathbb{O}_{y_jx_j\cdots\otimes y_kx_k\cdots}} & & \uparrow{\scriptstyle \mathbb{O}_0} \\
\hat{\mathcal{S}}(\emptyset) & \xrightarrow{C_i} & \hat{\mathcal{S}}(B_i) & \underset{m_i^{jk}}{\overset{\Delta^i_{jk}}{\rightleftarrows}} & \hat{\mathcal{S}}(B_j,B_k) & \xleftarrow{R_{jk}} & \hat{\mathcal{S}}(\emptyset) \\
\sqcup\to\cdot & & \circlearrowleft S_i & & & &
\end{array}$$

PBW Bases. The U's we care about always have "Poincaré-Birkhoff-Witt" bases; there is some finite set B and isomorphisms $\mathbb{O}_{y,x,\dots}\colon \hat{\mathcal{S}}(B)\to U$ defined by "ordering monomials" to some fixed $y,x,\dots$ order. The quantum group portfolio now becomes a "symmetric algebra" potfolio, or a "power series" portfolio.

The Category $\mathcal{DO}$. Hence we care about the monoidal category $\mathcal{DO}$ whose objects are finite sets B and whose morphisms are $\operatorname{mor}_{\mathcal{DO}}(B,B') := \operatorname{Hom}_{\mathbb{Q}}(\mathcal{S}(B)\to\mathcal{S}(B')) = \mathcal{S}(B^*,B')$ (by convention, $x^*=\xi$, $y^*=\eta$, etc.).

The Composition Law. If

$$\mathcal{S}(B_0)\ \xrightarrow[{}^t f\in\mathbb{Q}[\![\zeta_{0i},z_{1j}]\!]]{f}\ \mathcal{S}(B_1)\ \xrightarrow[{}^t g\in\mathbb{Q}[\![\zeta_{1j},z_{2k}]\!]]{g}\ \mathcal{S}(B_2)$$

then ${}^t(f/\!\!/g) = {}^t(g\circ f) = \left(g|_{\zeta_{1j}\to\partial_{z_{1j}}} f\right)_{z_{1j}=0}$ = (write also in alt. form)

Proposition. If $F\colon\mathcal{S}(B)\to\mathcal{S}(B')$ is linear and "continuous", then ${}^tF = \exp\left(\sum_{z_i\in B}\zeta_i z_i\right)/\!\!/F$.

Examples.

1. The 1-variable identity map $I\colon\mathcal{S}(z)\to\mathcal{S}(z)$ is given by ${}^tI_1 = \mathbb{e}^{z\zeta}$ and the n-variable one by ${}^tI_n = \mathbb{e}^{z_1\zeta_1+\cdots+z_n\zeta_n}$.
2. The "$z_i\to z_j$ variable rename map $\sigma^i_j\colon\mathcal{S}(z_i)\to\mathcal{S}(z_j)$ becomes ${}^t\sigma^i_j = \mathbb{e}^{z_j\zeta_i}$, and it's easy to rename several variables simultaneously.
3. The "archetypal multiplication map $m^{ij}_k\colon\mathcal{S}(z_i,z_j)\to\mathcal{S}(z_k)$" has ${}^tm = \mathbb{e}^{z_k(\zeta_i+\zeta_j)}$.
4. The "archetypal coproduct $\Delta^i_{jk}\colon\mathcal{S}(z_i)\to\mathcal{S}(z_j,z_k)$", given by $z_i\to z_j+z_k$ or $\Delta z = z\otimes 1+1\otimes z$, has ${}^t\Delta = \mathbb{e}^{(z_j+z_k)\zeta_i}$.
5. R-matrices tend to have terms of the form $\mathbb{e}_q^{\hbar y_1x_2}\in\mathcal{U}_q\otimes\mathcal{U}_q$. The "baby R-matrix" is ${}^tR = \mathbb{e}^{\hbar yx}\in\mathcal{S}(y,x)$.
6. The "Weyl form of the canonical commutation relations" states that if $[y,x]=t$ is a scalar, then $\mathbb{e}^{\xi x}\mathbb{e}^{\eta y} = \mathbb{e}^{\eta y}\mathbb{e}^{\xi x}\mathbb{e}^{-\eta\xi t}$. Thus with

$$SW_{xy}\circlearrowright\ \mathcal{S}(y,x)\ \underset{\mathbb{O}_{yx}}{\overset{\mathbb{O}_{xy}}{\rightrightarrows}}\ \mathcal{U}(y,x)$$

we have ${}^tSW_{xy} = \mathbb{e}^{\eta y+\xi x-\eta\xi t}$.

grey out.

Include from Metamonle as below.
Then
5. A graphid proof of zipping

"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

www.katlas.org The Knot Atlas

The Zipping Issue (between unbound and bound lies half-zipped).

1

Zipping. If $P(\zeta^j, z_i)$ is a polynomial, or whenever otherwise convergent, set

$$\left\langle P(\zeta^j, z_i)\right\rangle_{(\zeta^j)} = P\left(\partial_{z_j}, z_i\right)\Big|_{z_i=0}.$$

(E.g., if $P = \sum a_{nm}\zeta^n z^m$ then $\langle P\rangle_\zeta = \sum n! a_{nn}$).

2

The Zipping / Contraction Theorem. If P has a finite ζ-degree and the y's and the q's are "small" then

$$\left\langle P(z_i, \zeta^j) e^{\eta^i z_i + y_j \zeta^j}\right\rangle_{(\zeta^j)} = \left\langle P(z_i + y_i, \zeta^j) e^{\eta^i (z_i + y_i)}\right\rangle_{(\zeta^j)},$$

(proof: replace $y_j \to \hbar y_j$ and test at $\hbar = 0$ and at $\partial_\hbar$), and

$$\left\langle P(z_i, \zeta^j) e^{c + \eta^i z_i + y_j \zeta^j + q^i_j z_i \zeta^j}\right\rangle_{(\zeta^j)} = \det(\tilde{q}) \left\langle P(\tilde{q}^k_i (z_k + y_k), \zeta^j) e^{c + \eta^i \tilde{q}^k_i (z_k + y_k)}\right\rangle_{(\zeta^j)}$$

where $\tilde{q}$ is the inverse matrix of $1 - q$: $(\delta^i_j - q^i_j)\tilde{q}^j_k = \delta^i_k$ (proof: replace $q^i_j \to \hbar q^i_j$ and test at $\hbar = 0$ and at $\partial_\hbar$).

4

Implementation. ωεβ/ZipBindDemo

```
Kδ /: Kδ{i_,j_} := If[i === j, 1, 0];
{z*, x*, y*} = {ζ, ξ, η}; {ζ*, ξ*, η*} = {z, x, y};
(u_i_)* := (u*)i;
```

```
Zip{}[P_] := P;
Zip{ζ_,ζs___}[P_] :=
  (Expand[P // Zip{ζs}] /. f_. ζ^d_. :> ∂{ζ*,d} f) /. ζ* → 0
Zip{ζ}[(a ζ^6 + ζ + 3) (z^5 e^z + 7 z) + 99 b]
7 + 720 a + 99 b
Zip{ξ,η}[ξ^3 η^3 e^(a x + b y + c x y)]
a^3 b^3 + 9 a^2 b^2 c + 18 a b c^2 + 6 c^3
```

3 w/ better examples

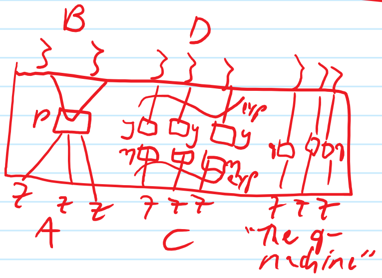
5. A graphical proof

The true Hilbert hotel is exp! Remove one x from an "exponential reservoir" of x's and you are left w/ the same exponential reservoir:

$$\exp x = \left[\dots + \frac{xxxxx}{120} + \dots\right] \xrightarrow{\partial_x} \left[\dots + \frac{\cancel{x}xxxx}{120} + \dots\right] = (\exp x)' = \exp x$$

And if you let each element choose left or right, you get twice the same reservoir:

$$e^x \xrightarrow{x \mapsto x_l + x_r} e^{x_l + x_r} = e^{x_l} e^{x_r}$$



Several scenarios occur:

1. Start at A, go through the q machine $k \geq 0$ times, stop at B:

$$\langle P(\sum_{k\geq 0} q^k z, \zeta)\rangle = \langle P(\tilde{q} z, \zeta)\rangle$$

2. Loop through the q machine & swallow your own tail:

$$\exp\left(\sum \frac{q^k}{k}\right) = \exp(-\log(1-q)) = \tilde{q}$$

3. ......

By the room splitting principle, these scenario contribute multiplicatively.

Sketch. (Total 57m)

1. (5m) The knot theory portfolio.
2. (5m) The (quantum) group portfolio. (Write in general-algebra language, not in universal enveloping algebra language).
3. (3m) Quantum groups have "PBW" basis. Hence they are symmetric algebras, with funny products / co-products.
4. (3m) The DO category.
5. (5m) Example: the Abelian case.
6. (3m) Example: k=0.
7. (5m) Example: sl2.
8. (10m) Zipping and composing. The zipping theorem.
9. (5m) Docility and polynomiallity.
10. (5m) Solvable Approximation.
11. (8m) Mention Rozansky-Overbay, mention computations.

References.

[BV1] D. Bar-Natan and R. van der Veen, *A Polynomial Time Knot Polynomial,* arXiv:1708.04853.

[Ov] A. Overbay, *Perturbative Expansion of the Colored Jones Polynomial,* University of North Carolina PhD thesis, ωεβ/Ov.

[Ro1] L. Rozansky, *A contribution of the trivial flat connection to the Jones polynomial and Witten's invariant of 3d manifolds, I,* Comm. Math. Phys. **175-2** (1996) 275–296, arXiv:hep-th/9401061.

[Ro2] L. Rozansky, *The Universal R-Matrix, Burau Representation and the Melvin-Morton Expansion of the Colored Jones Polynomial,* Adv. Math. **134-1** (1998) 1–31, arXiv:q-alg/9604005.

[Ro3] L. Rozansky, *A Universal* $U(1)$*-RCC Invariant of Links and Rationality Conjecture,* arXiv:math/0201139.