Title: What else can you do with solvable approximations?

Date: Mar 12, 2018 02:00 PM

URL: http://pirsa.org/18030082

Abstract: <p>Recently, Roland van der Veen and myself found that there are sequences of solvable Lie algebras "converging" to any given semi-simple Lie algebra (such as sl(2) or sl(3) or E8). Certain computations are much easier in solvable Lie algebras; in particular, using solvable approximations we can compute in polynomial time certain projections (originally discussed by Rozansky) of the knot invariants arising from the Chern-Simons-Witten topological quantum field theory. This provides us with the first strong knot invariants that are computable for truly large knots. But sl(2) and sl(3) and similar algebras occur in physics (and in mathematics) in many other places, beyond the Chern-Simons-Witten theory. Do solvable approximations have further applications?

This is a repeat of a talk I gave in McGill University in February, 2017. A video recording, a handout, and some further links are at McGill-1702</p>

Dror Bar-Natan: Talks: McGill-1602:

Joint with Roland van der Veen

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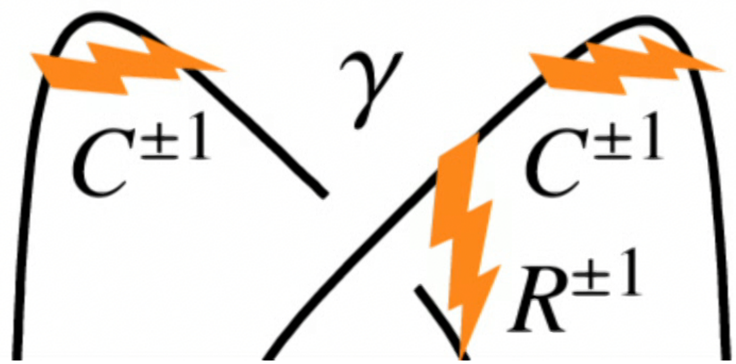
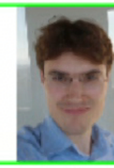


Dror Bar-Natan: Talks: GWU-1612:

On Elves and Invariants





Follows Rozan
Overbay [Ov], j

Abstract. Whether or not you like the formulas on this page, they describe the strongest truly computable knot invariant we know.

Three steps to the computation of ρ_1:

1. Preparation. Given K, results

$$\langle \textit{long word} \,\|\, \textit{simple formulas} \rangle.$$

2. Rewrite rules. Make the *word* simpler and the *formulas* more complica-

Knot K

$\downarrow$ preparation

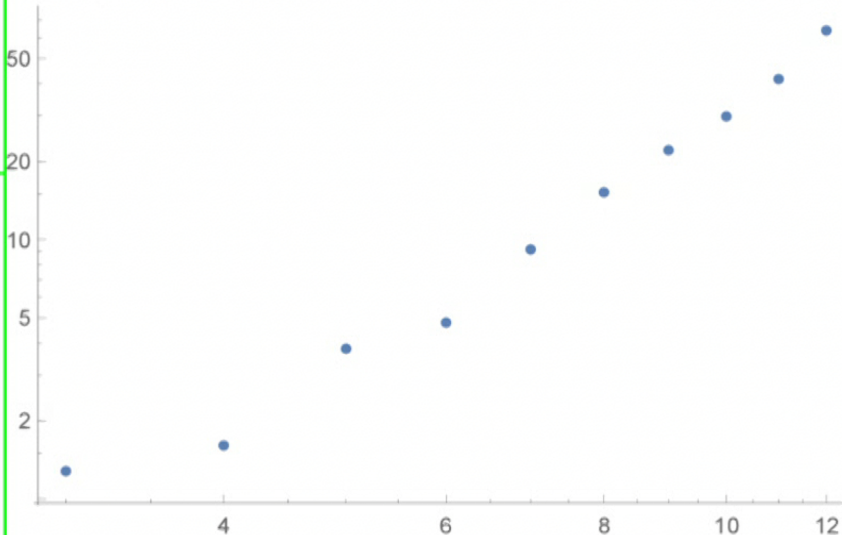
$$\langle elf \ldots elf \,\|\, \omega_0; L_0; Q_0; P_0 \rangle$$

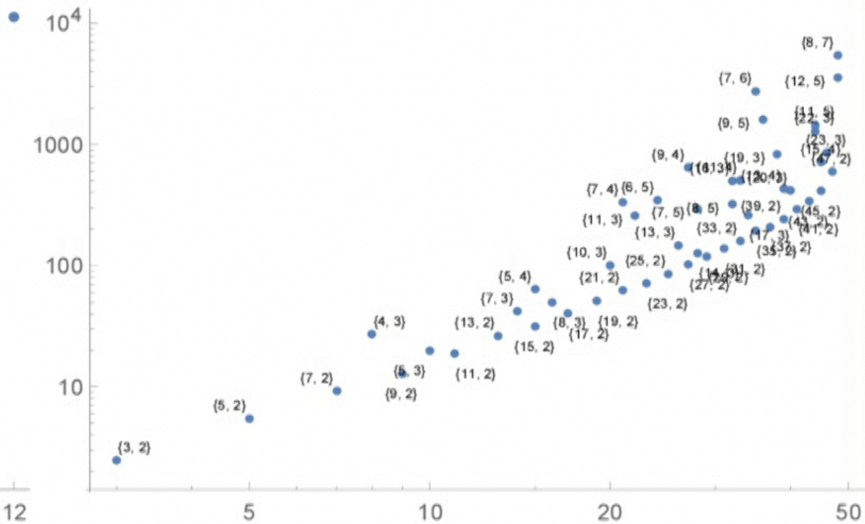
$\downarrow$ rewrite rules

$$\rho_1(K) := \frac{\cdots}{(t-1)^2\omega^2}.$$

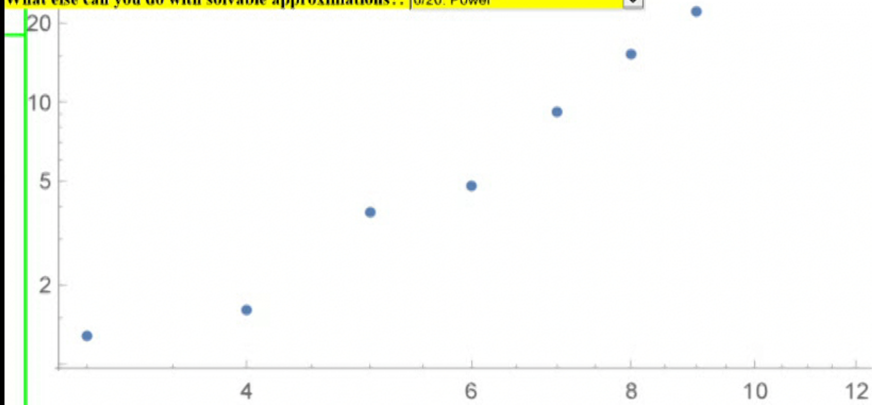
(ω is the Alexander polynomial, L and Q are not interesting).

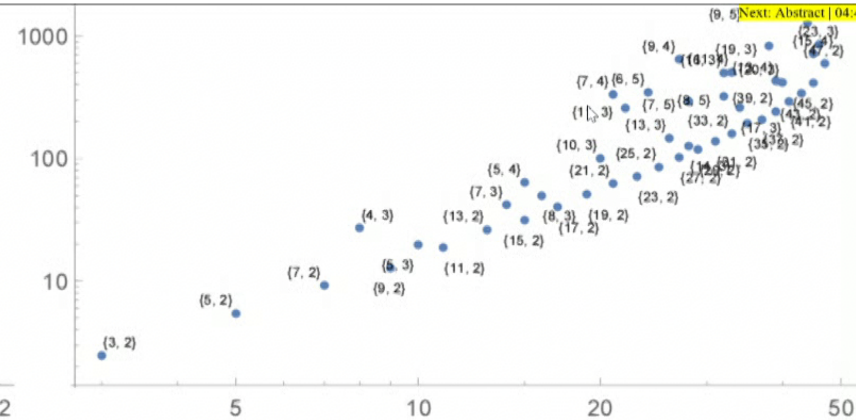
Experimental Analysis (ωεβ/Exp). Log-log plots of computation time (sec) vs. crossing number, for all knots with up to 12 crossings (mean times) and for all torus knots with up to 48 crossings:





Power. On the 250 knots with at most 10 crossings, the pair (ω, ρ_1) attains 250 distinct values, while (Khovanov, HOMFLY-





Power. On the 250 knots with at most 10 crossings, the pair (ω, ρ_1) attains 250 distinct values, while (Khovanov, HOMFLY-PT) attains only 249 distinct values. To 11 crossings the numbers are (802, 788, 772) and to 12 they are (2978, 2883, 2786).

Genus. Up to 12 xings, always ρ_1 is symmetric under $t \leftrightarrow t^{-1}$. With ρ_1^+ denoting the positive-degree part of ρ_1, always $\deg \rho_1^+ \leq 2g - 1$, where g is the 3-genus of K (equallity for 2530 knots). This gives a lower bound on g in terms of ρ_1 (conjectural, but undoubtedly true). This bound is often weaker than the Alexander

What else can you do with solvable approximations

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But sl_2 and sl_3 and similar algebras occur in physics (and in mathematics) in many other places, beyond the Chern-Simons-Witten theory. Do solvable approximations have further applications?

Recomposing gl_n. Half is enough! $gl_n \oplus \mathfrak{a}_n = \mathcal{D}(\triangledown, b, \delta)$:

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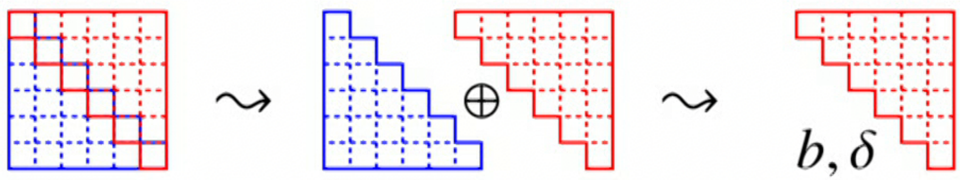
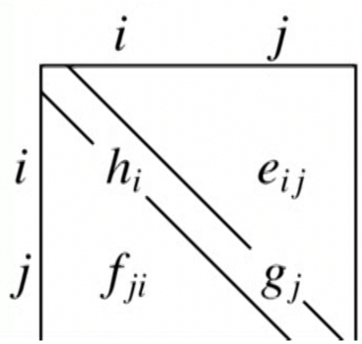
Recomposing gl_n. Half is enough! $gl_n \oplus \mathfrak{a}_n = \mathcal{D}(◹, b, \delta)$:



$b(◹) = b$: ◹ ⊗ ◹ → ◹
$b(◺) \rightsquigarrow \delta$: ◹ → ◹ ⊗ ◹

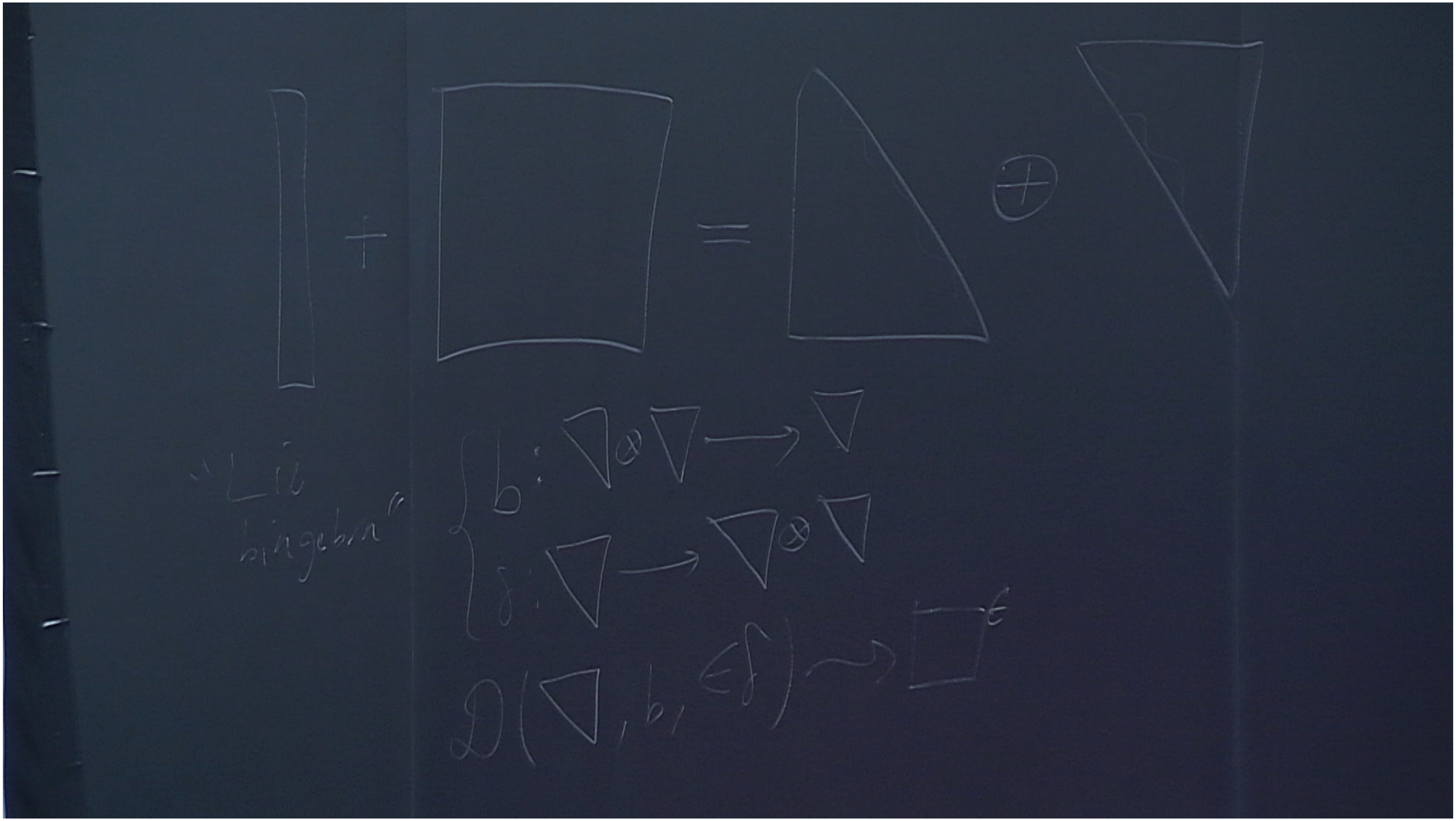
Now define $gl_n^\epsilon := \mathcal{D}(◹, b, \epsilon\delta)$. Schematically, this is [◹, ◹] = ◹, [◺, ◺] = ϵ◺, and [◹, ◺] = ◺ + ϵ◹. In detail, it is

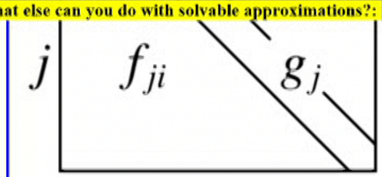


$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{li} e_{kj} \qquad [f_{ij}, f_{kl}] = \epsilon\delta_{jk} f_{il} - \epsilon\delta_{li} f_{kj}$$

$$[e_{ij}, f_{kl}] = \delta_{jk}(\epsilon\delta_{j<k} e_{il} + \delta_{il}(h_i + \epsilon g_i)/2 + \delta_{i>l} f_{il}) - \delta_{li}(\epsilon\delta_{k<j} e_{kj} + \delta_{kj}(h_j + \epsilon g_j)/2 + \delta_{k>j} f_{kj})$$

$$[g_i, e_{jk}] = (\delta_{ij} - \delta_{ik}) e_{jk} \qquad [h_i, e_{jk}] = \epsilon(\delta_{ij} - \delta_{ik}) e_{jk}$$


"Lie bialgebra"
b:
δ:
ε



$$-\delta_{li}(\epsilon\delta_{k<j}e_{kj} + \delta_{kj}(h_j + \epsilon g_j)/2 + \delta_{k>j}f_{kj})$$

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Solvable Approximation. At $\epsilon = 1$ and modulo $h = g$, the above is just gl_n. By rescaling at $\epsilon \neq 0$, gl_n^ϵ is independent of ϵ. We let gl_n^k be gl_n^ϵ regarded as an algebra over $\mathbb{Q}[\epsilon]/\epsilon^{k+1} = 0$. It is the "$k$-smidgen solvable approximation" of gl_n!

Recall that $\mathfrak{g}$ is "solvable" if iterated commutators in it ultimately vanish: $\mathfrak{g}_2 := [\mathfrak{g}, \mathfrak{g}]$, $\mathfrak{g}_3 := [\mathfrak{g}_2, \mathfrak{g}_2]$, $\dots$, $\mathfrak{g}_d = 0$. Equivalently, if it is a subalgebra of some large-size $\triangledown$ algebra.

Note. This whole process makes sense for arbitrary semi-simple Lie algebras.

Why are "solvable algebras" any good? Contrary to common beliefs, computations in semi-simple Lie algebras are just awful:

...ss makes sense for arbitrary semi-...

Lie algebras.

Why are "solvable algebras" any good? Contrary to common beliefs, computations in semi-simple Lie algebras are just awful:

In[1]:= MatrixExp$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right]$ // FullSimplify // MatrixForm *Enter*

Yet in solvable algebras, exponentiation is fine and even BCH, $z = \log(e^x e^y)$, is bearable:

In[2]:= MatrixExp$\left[\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right]$ // MatrixForm

Out[2]//MatrixForm=

$$\begin{pmatrix} e^a & \frac{b\left(e^a - e^c\right)}{a-c} \\ 0 & e^c \end{pmatrix}$$

In[3]:= MatrixExp$\left[\begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}\right]$.MatrixExp$\left[\begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix}\right]$ // MatrixLog // PowerExpand // Simplify // MatrixForm *Enter*

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Out[1]//MatrixForm=

$$\left(\begin{matrix} \frac{e^{\frac{a+d}{2}}\left(\sqrt{4bc+(a-d)^2}\,\text{Cosh}\left[\frac{1}{2}\sqrt{4bc+(a-d)^2}\right]+(a-d)\,\text{Sinh}\left[\frac{1}{2}\sqrt{4bc+(a-d)^2}\right]\right)}{\sqrt{4bc+(a-d)^2}} & \\ \frac{2c\,e^{\frac{a+d}{2}}\,\text{Sinh}\left[\frac{1}{2}\sqrt{4bc+(a-d)^2}\right]}{\sqrt{4bc+(a-d)^2}} & \frac{e^{\frac{a+d}{2}}\left(\sqrt{4bc+(a\ldots\right.}{} \end{matrix}\right.$$

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                    c  d )] // FullSimplify // MatrixForm
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Enter

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$$\begin{pmatrix} e^a & \frac{b\left(e^a - e^c\right)}{a-c} \\ 0 & e^c \end{pmatrix}$$

```
In[3]:= MatrixExp[( a1  b1
                    0   c1 )].MatrixExp[( a2  b2
                                          0   c2 )] //
          MatrixLog // PowerExpand // Simplify //
        MatrixForm
```

Enter

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Out[3]//MatrixForm=

$$\begin{pmatrix} a_1 + a_2 & \frac{(a_1+a_2-c_1-c_2)\left(e^{c_2}\left(e^{a_1}-e^{c_1}\right) a_2\, b_1+e^{a_1}\left(e^{a_2}-e^{c_2}\right) a_1\, b_2-e^{a_1+a_2}\, b_2\, c_1+e^{a_1+c}\cdots\right.}{\left(e^{a_1+a_2}-e^{c_1+c_2}\right)(a_1-c_1)(a_2-c_2)} \\ 0 & c_1 + c_2 \end{pmatrix}$$

In[3]:= $\text{MatrixExp}\left[\begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}\right].\text{MatrixExp}\left[\begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix}\right]$ // MatrixLog // PowerExpand // Simplify // MatrixForm

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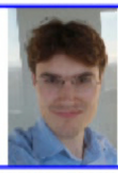

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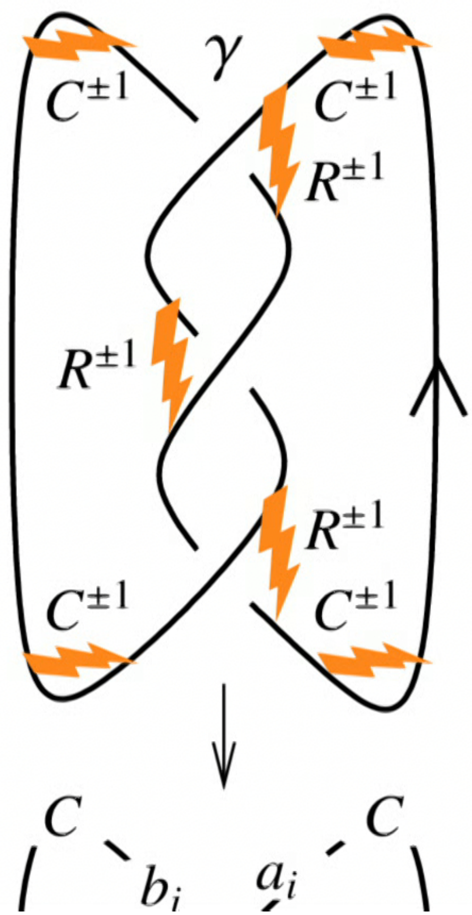
Chern-Simons-Witten. Given a knot $\gamma(t)$ in $\mathbb{R}^3$ and a metrized Lie algebra $\mathfrak{g}$, set $Z(\gamma) :=$

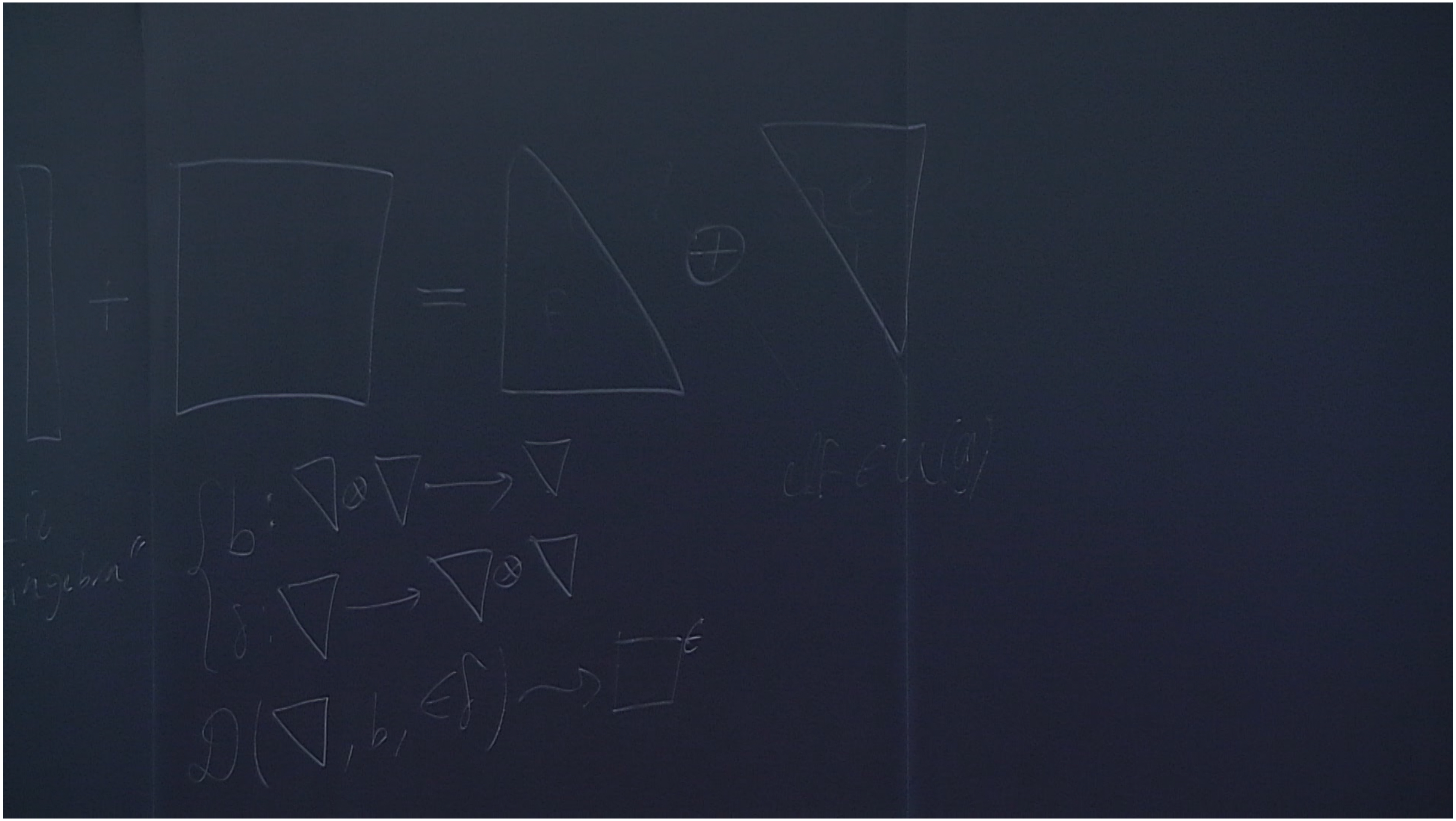
$$\int_{A\in\Omega^1(\mathbb{R}^3,\mathfrak{g})} \mathcal{D}A\, e^{ik\,cs(A)} PExp_\gamma(A),$$

where $cs(A) := \frac{1}{4\pi}\int_{\mathbb{R}^3} \operatorname{tr}\left(AdA + \frac{2}{3}A^3\right)$ and

$$PExp_\gamma(A) := \prod_0^1 \exp(\gamma^* A) \in \mathcal{U} = \hat{\mathcal{U}}(\mathfrak{g}),$$

and $\mathcal{U}(\mathfrak{g}) := \langle\text{words in } \mathfrak{g}\rangle/(xy - yx = [x,y])$. In a favourable gauge, one may hope that this computation will localize near the crossings and the bends, and all will depend on just two





$$(1+x_1)(1+x_2)\cdots\cdots(1+x_n)$$

$$1+\sum x_i+\sum_{i<j} x_i x_j+\cdots$$

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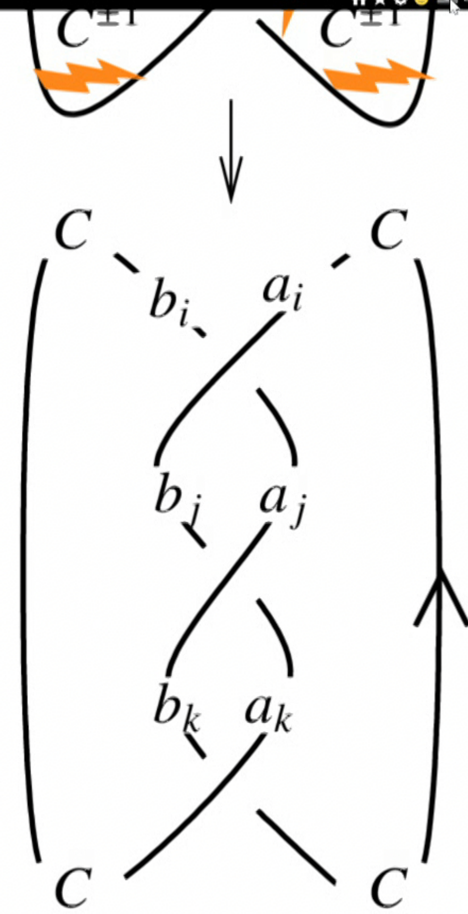
$$R = \sum a_i \otimes b_i \in \mathcal{U} \otimes \mathcal{U} \quad \text{and} \quad C \in \mathcal{U}.$$

This was never done formally, yet R and C can be "guessed" and all "quantum knot invariants" arise in this way. So for the trefoil,

$$Z = \sum_{i,j,k} C a_i b_j a_k C^2 b_i a_j b_k C.$$



But Z lives in $\mathcal{U}$, a complicated space. How do you extract information out of it?

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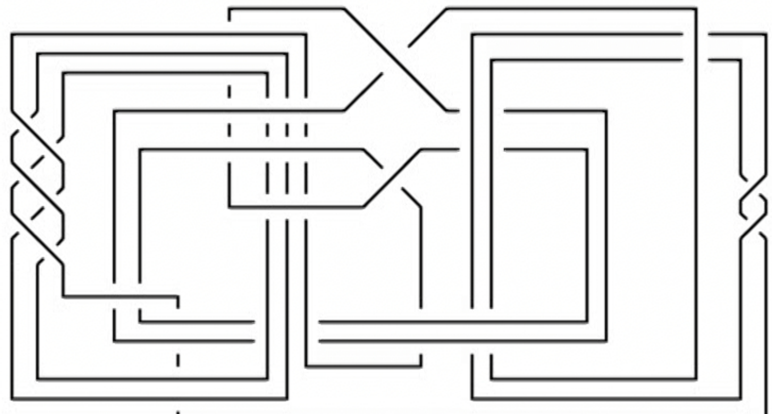
But Z lives in $\mathcal{U}$, a complicated space. How do you extract information out of it?

Solution 1, Representation Theory. Choose a finite dimensional representation ρ of $\mathfrak{g}$ in some vector space V. By luck and the wisdom of Drinfel'd and Jimbo, $\rho(R) \in V^* \otimes V^* \otimes V \otimes V$ and $\rho(C) \in V^* \otimes V$ are computable, so Z is computable too. But in exponential time!

Ribbon=Slice?

Gompf, Scharlemann, Thompson



Solution 2, Solvable Approximation. Work directly in $\hat{\mathcal{U}}(\mathfrak{g}_k)$, w-

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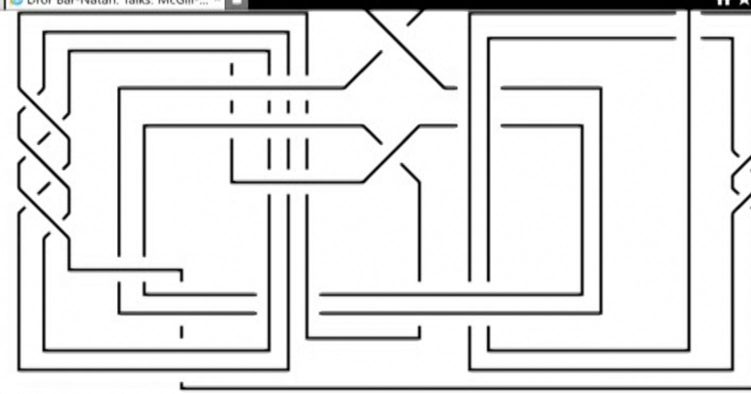

Solution 2, Solvable Approximation. Work directly in $\hat{\mathcal{U}}(\mathfrak{g}_k)$, where $\mathfrak{g}_k = sl_2^k$ (or a similar algebra); everything is expressible using low-degree polynomials in a small number of variables, hence everything is poly-time computable!

Example 0. Take $\mathfrak{g}_0 = sl_2^0 = \mathbb{Q}\langle h, e, l, f\rangle$, with h central and $[f,l] = f$, $[e,l] = -e$, $[e,f] = h$. In it, using normal orderings,

$$R = \mathbb{O}\left(\exp\left(hl + \frac{e^h - 1}{h} ef\right) \mid e \otimes lf\right), \quad \text{and,}$$

$$\mathbb{O}\left(e^{\delta ef} \mid fe\right) = \mathbb{O}\left(\nu e^{\nu\delta ef} \mid ef\right) \quad \text{with } \nu = (1 + h\delta)^{-1}$$

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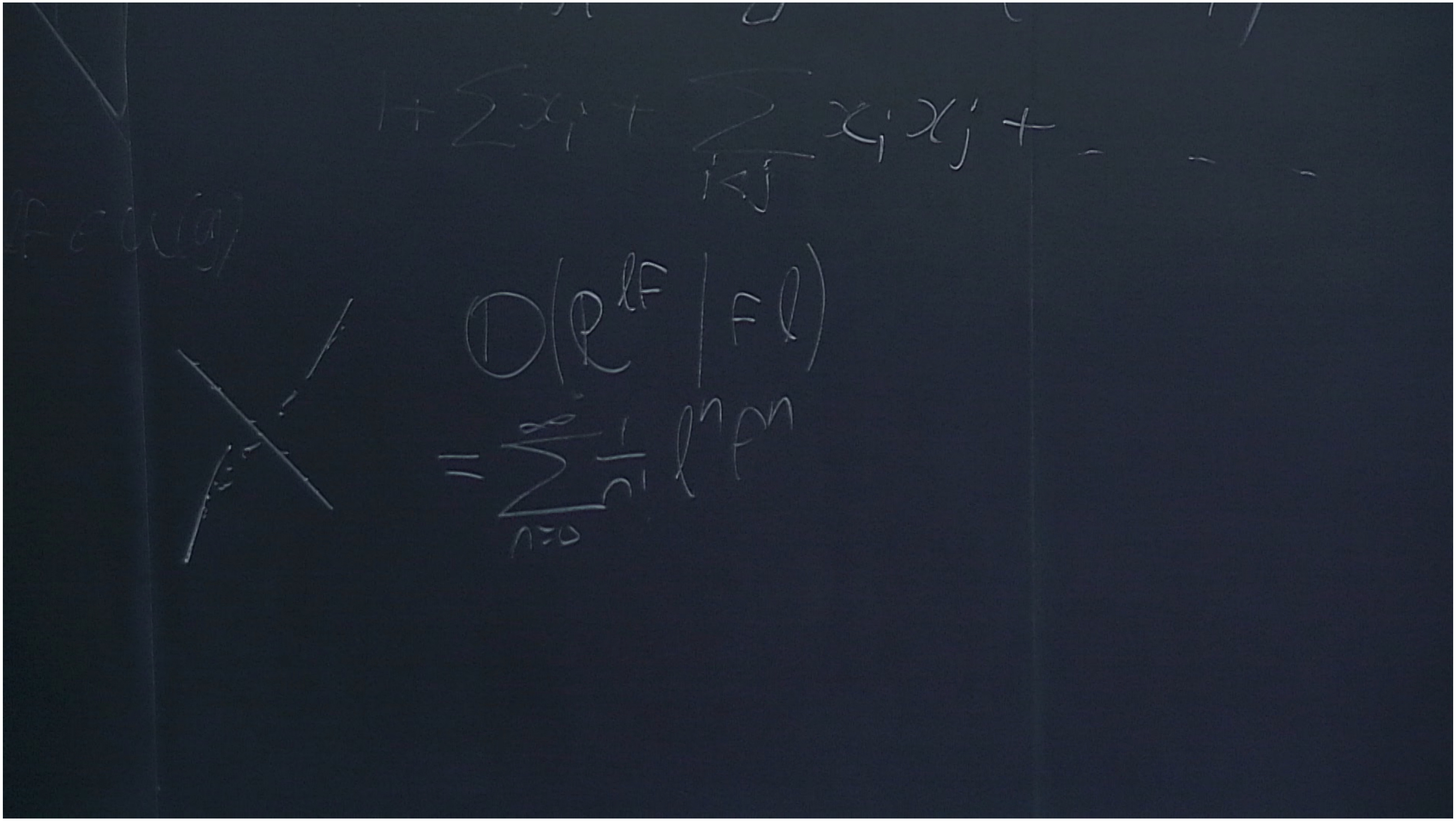
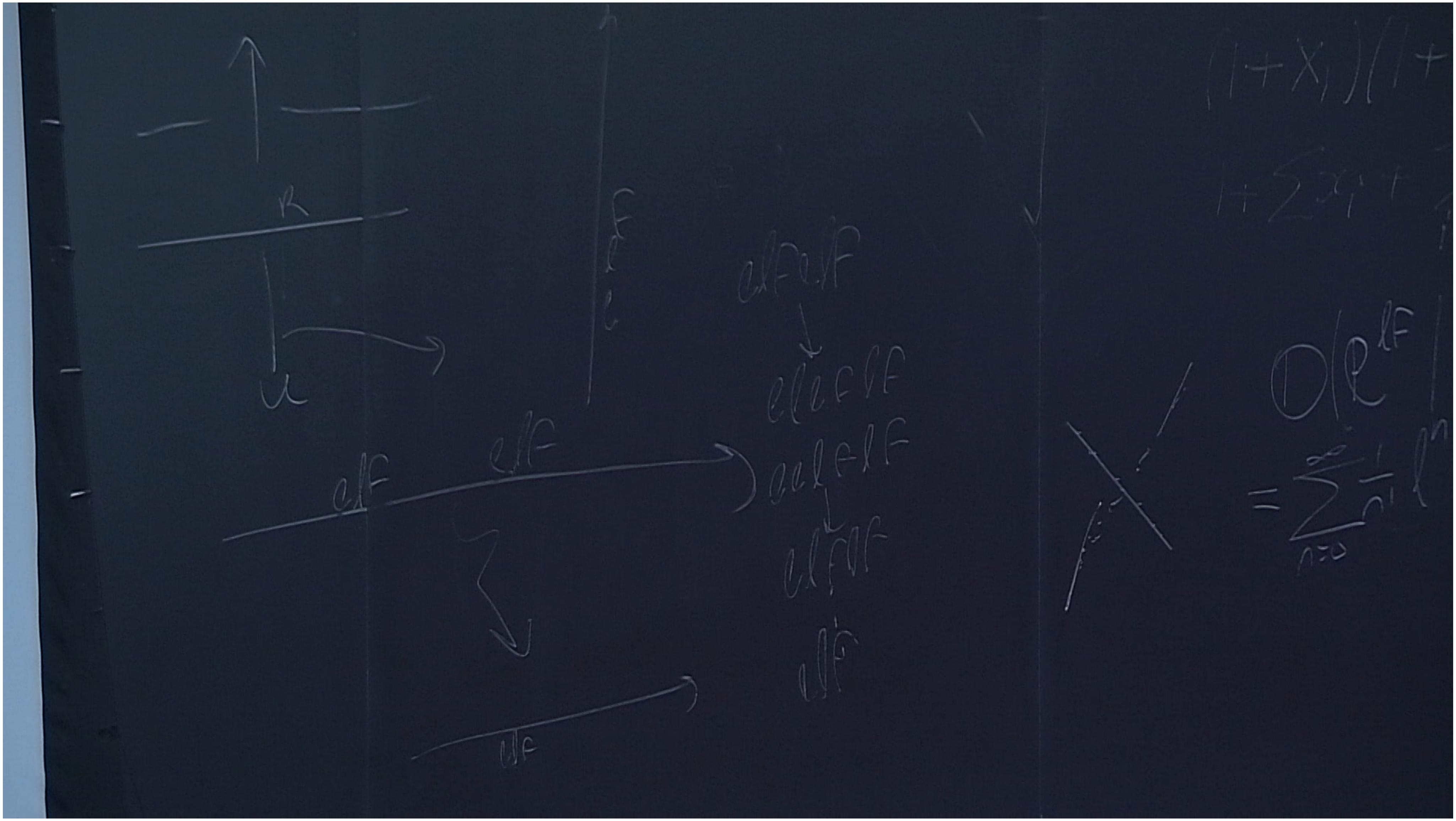
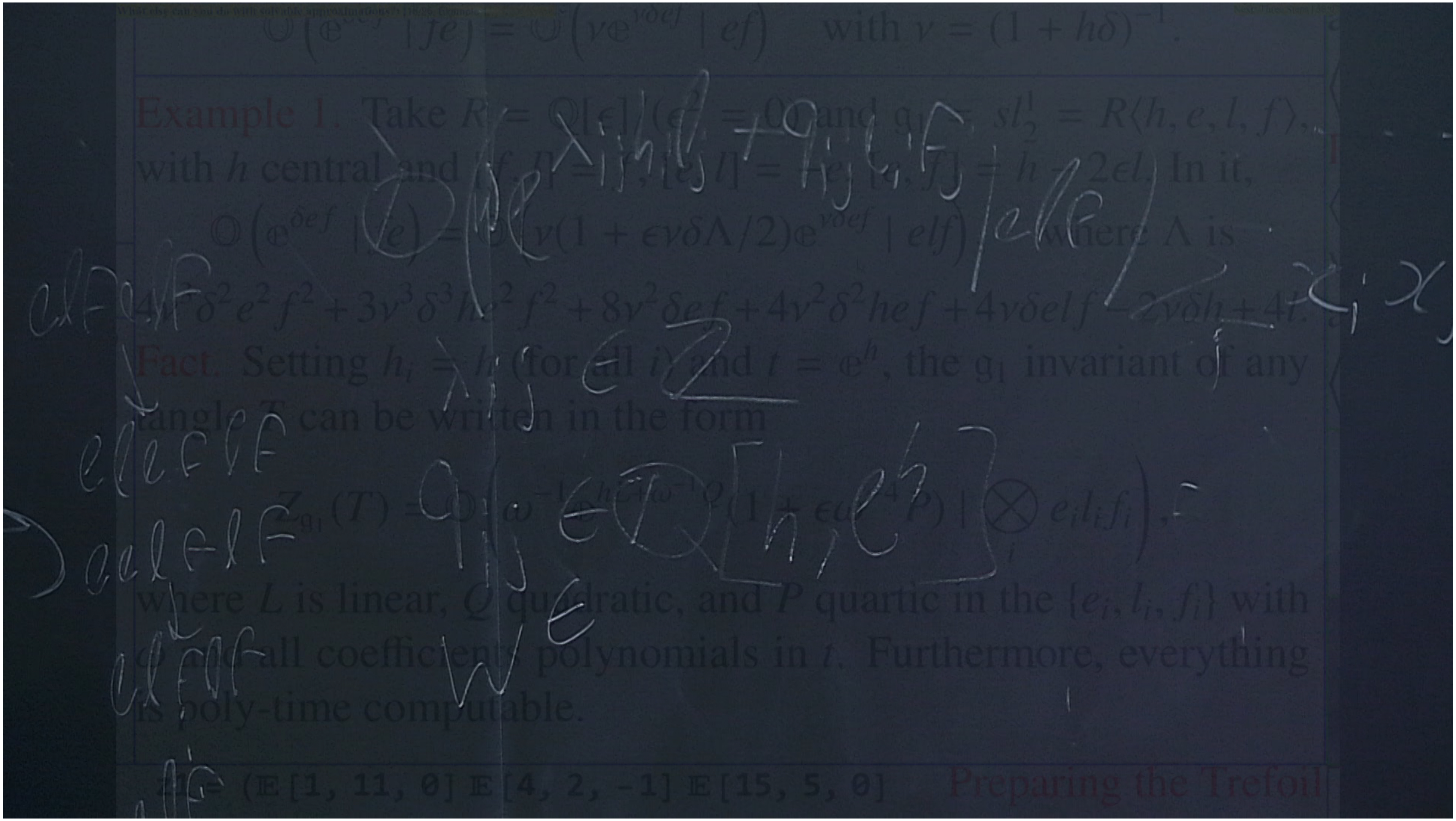
Example 1. Take $R = \mathbb{Q}[\epsilon]/(\epsilon^2 = 0)$ and $\mathfrak{g}_1 = sl_2^1 = R\langle h, e, l, f\rangle$, with h central and $[f, l] = f$, $[e, l] = -e$, $[e, f] = h - 2\epsilon l$. In it,

$$\mathbb{O}\left(\mathbb{e}^{\delta ef} \mid fe\right) = \mathbb{O}\left(\nu(1 + \epsilon\nu\delta\Lambda/2)\mathbb{e}^{\nu\delta ef} \mid elf\right), \quad \text{where } \Lambda \text{ is}$$

$4\nu^3\delta^2e^2f^2 + 3\nu^3\delta^3he^2f^2 + 8\nu^2\delta ef + 4\nu^2\delta^2hef + 4\nu\delta elf - 2\nu\delta h + 4l$


$1 + \sum_i x_i + \sum_{i<j} x_i x_j + \ldots$
$\mathbb{O}(e^{lF} | F l)$
$= \sum_{n=0}^{\infty} \frac{1}{n!} l^n e^n$

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$\mathbb{O}\left(e^{\delta ef} \mid le\right) = \mathbb{O}\left(\nu(1 + \epsilon\nu\delta\Lambda/2)e^{\nu\delta ef} \mid elf\right)$ where Λ is
$4\nu^3\delta^2e^2f^2 + 3\nu^3\delta^3he^2f^2 + 8\nu^2\delta ef + 4\nu^2\delta^2hef + 4\nu\delta elf - 2\nu\delta h + 4l$.
Fact. Setting $h_i = h$ (for all i) and $t = e^h$, the $\mathfrak{g}_1$ invariant of any tangle T can be written in the form
$Z_{\mathfrak{g}_1}(T) = \mathbb{O}\left(\omega^{-1}e^{hL+\omega^{-1}Q}(1 + \epsilon\omega^{-4}P) \mid \bigotimes_i e_il_if_i\right)$,
where L is linear, Q quadratic, and P quartic in the $\{e_i, l_i, f_i\}$ with ω and all coefficients polynomials in t. Furthermore, everything is poly-time computable.
Z = (𝔼[1, 11, 0] 𝔼[4, 2, -1] 𝔼[15, 5, 0]
Preparing the Trefoil

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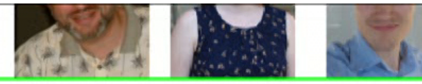
Preparing the Trefoil

On Elves and Invariants



Overbay [OV], J

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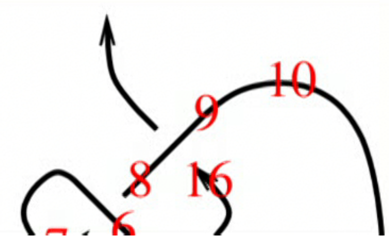
$$\langle \textit{long word} \,\|\, \textit{simple formulas} \rangle.$$

2. Rewrite rules. Make the *word* simpler and the *formulas* more complicated, until the *word* "*elf*" is reached.

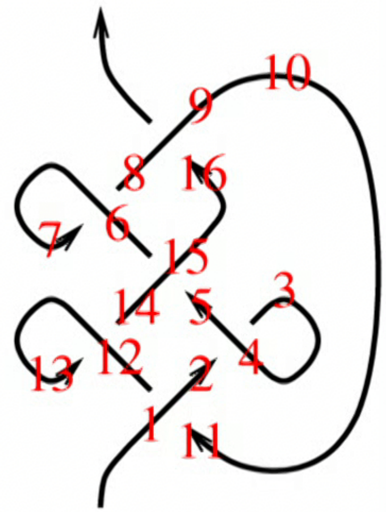
3. Readout. The invariant ρ_1 is read from the last *formulas*.

$$\text{Knot } K$$
$$\downarrow \text{ preparation}$$
$$\langle \textit{elf} \ldots \textit{elf} \,\|\, \omega_0; L_0; Q_0; P_0 \rangle$$
$$\downarrow \text{ rewrite rules}$$
$$\langle \textit{elf} \,\|\, \omega; -; -; P \rangle$$
$$\downarrow \text{ readout}$$
$$\rho_1(K) = \rho_1(\omega, P)$$

Preparation. Draw K using a 0-framed 0-rotation planar diagram D where all crossings are pointing up. Walk along D labeling features by



Preparation. Draw K using a 0-framed 0-rotation planar diagram D where all crossings are pointing up. Walk along D labeling features by $1,\dots,m$ in order: over-passes, under-passes, and right-heading cups and caps ("±-cuaps"). If x is a xing, let i_x and j_x be the labels on its over/under strands, and let s_x be 0 if it right-handed and -1 otherwise. If c is a cuap, let i_c be its label and s_c be its sign. Set



$$(L;Q;P) = \sum_{x:\,(i,j,s)} (-)^s \left(l_j;\ t^s e_i f_j;\ (-t)^s e_i l_{(1+s)i-sj} f_j + l_i l_j + \frac{t^{2s} e_i^2 f_j^2}{4} \right) + \sum_{c:\,(i,s)} (0;\ 0;\ s\cdot l_i)\,.$$

This done, output $\langle e_1 l_1 f_1 e_2 l_2 f_2 \cdots e_m l_m f_m \,\|\, 1; L; Q; P\rangle$.

Rule 5, *fe* Sorts. Provided k introduces no clashes, given $\langle \dots f_i e_j \dots \| \omega; L; Q; P\rangle$, decompose $Q = Q_{fe} f_i e_j + Q_f f_i + Q_e e_j + Q'$ write $P = P(f_i, e_j)$ (with messy coefficients), set $\mu = 1+(t-1)\delta$ and $q = ((1-t)\alpha\beta + \beta e_k + \alpha f_k + \delta e_k f_k)/\mu$, and output

$$\left\langle \dots e_k f_k \dots \middle\| \begin{matrix} \mu\omega; L; \mu\omega q + \mu Q'; \\ \omega^4 \Lambda_k + \mathbb{e}^{-q} P(\partial_\alpha, \partial_\beta)(\mathbb{e}^q) \end{matrix} \right\rangle \Bigg|_{\substack{\alpha\to Q_f/\omega, \beta\to Q_e/\omega, \\ \delta \to Q_{fe}/\omega}},$$

where Λ_k is the Λόγος, "a principle of order and knowledge":

$$\begin{aligned}\Lambda_k = \frac{t+1}{4}\Big(& -\delta(\mu+1)\left(\beta^2 e_k^2 + \alpha^2 f_k^2\right) - \delta^3(3\mu+1)e_k^2 f_k^2 \\ & - 2(\beta e_k + \alpha f_k)\left(\alpha\beta + 2\delta\mu + \delta^2(2\mu+1)e_k f_k + 2\delta\mu^2 l_k\right) \\ & - 4(\alpha\beta + \delta\mu)\left(\delta(\mu+1)e_k f_k + \mu^2 l_k\right) - 4\delta^2\mu^2 e_k f_k l_k \\ & + (t-1)\left(2(\alpha\beta+\delta\mu)^2 - \alpha^2\beta^2\right)\Big).\end{aligned}$$

What else can you do with solvable approximations?: 23/26. Rolfsen table

Next: E&I Abstract, again | 1:02:17

$$-\frac{e_4 f_2}{t} + e_{15} f_5 - \frac{e_6 f_8}{t} + e_1 f_{11} - \frac{e_{12} f_{14}}{t} + e_9 f_{16},$$

$$-\frac{e_4^2 f_2^2}{4t^2} + \frac{1}{4} e_{15}^2 f_5^2 - \frac{e_6^2 f_8^2}{4t^2} + \frac{1}{4} e_1^2 f_{11}^2 - \frac{e_{12}^2 f_{14}^2}{4t^2} + \frac{1}{4} e_9^2 f_{16}^2 + e_1 f_{11} l_1 + \frac{e_4 f_2 l_2}{t} - l_3 - l_2 l_4 + l_7 + \frac{e_6 f_8 l_8}{t} - l_6 l_8 + e_9 f_{16} l_9 - l_{10} + l_1 l_{11} + l_{13} - \frac{e_{12} f_{14} l_{14}}{t} - l_{12} l_{14} + e_{15} f_5 l_{15} + l_5 l_{15} + l_9 l_{16}\Big]$$

Differential Polynomials (Implementing $P(\partial_\alpha, \partial_\beta)(f)$)

```
DP[x_ -> D[α_], y_ -> D[β_]][P_][f_] :=
  Total[CoefficientRules[P, {x, y}] /.
    ({m_, n_} -> c_) :> c D[f, {α, m}, {β, n}]]
```

le and fl Sorts

```
S[l[j_], (x:e|f)[i_] -> k_][E[ω_, L_, Q_, P_]] :=
  With[{λ = ∂[l[j]] L, α = ∂[x[i]] Q, q = e^γ β x[k] + γ l[k]}, CF[
    E[ω, L /. l[j] -> l[k], t^λ α x[k] + (Q /. x[i] -> 0),
      e^-q DP[l[j] -> D[γ], x[i] -> D[β]][P][e^q] /. (β -> α/ω, γ -> λ Log[t])] ]];
```

The Λόγος

```
Λ[k_] := ((t - 1) (2 (α β + δ μ)^2 - α^2 β^2) - 4 e[k] l[k] f[k] δ^2 μ^2 -
     δ (1 + μ) (f[k]^2 α^2 + e[k]^2 β^2) - e[k]^2 f[k]^2 δ^3 (1 + 3 μ) -
     2 (α β + 2 δ μ + e[k] f[k] δ^2 (1 + 2 μ) + 2 l[k] δ μ^2) (f[k] α + e[k] β) -
     4 (l[k] μ^2 + e[k] f[k] δ (1 + μ)) (α β + δ μ)) (1 + t) / 4;
```

fe Sorts

```
S[f[i_] e[j_] -> k_][E[ω_, L_, Q_, P_]] :=
  With[{q = ((1 - t) α β + β e[k] + α f[k] + δ e[k] f[k]) / μ}, CF[
    E[μ ω, L, μ ω q + μ (Q /. f[i] | e[j] -> 0),
      μ^4 e^-q DP[f[i] -> D[α], e[j] -> D[β]][P][e^q] + ω^4 Λ[k]] /. μ -> 1 + (t - 1) δ /.
    {α -> ω^-1 (∂[f[i]] Q /. e[j] -> 0), β -> ω^-1 (∂[e[j]] Q /. f[i] -> 0),
     δ -> ω^-1 ∂[f[i], e[j]] Q}]];
```

Elf Merges

```
m[i_, j_ -> k_][Z_E] := Module[{x, z},
  CF[(Z // S[f[i] e[j] -> x] // S[l[i] e[x] -> x] // S[f[x] l[j] -> x]) /. z_[i|j|x] -> z[k]]]
```

Rewriting the Trefoil (by merging 16 elves)

```
(Do[z1 = z1 // m[1, k -> 1], {k, 2, 16}]; z1)
```

$$\mathbb{E}\Big[\frac{1-t+t^2}{t}, 0, 0, \frac{(-1+t)(1-t+t^2)^2(1-t+2t^2)}{t^3} - \frac{2(1+t)(1-t+t^2)^3 e_1 f_1}{t^4} - \frac{2(-1+t)(1+t)(1-t+t^2)^3 l_1}{t^4}\Big]$$

Readout

```
ρ1[E[ω_, _, _, P_]] := CF[ t ((P /. e | l | f -> 0) - t ω^3 (∂t ω)) / ((t - 1)^2 ω^2) ]
```

$\rho_1(3_1)$

```
ρ1[z1] // Expand
```

$\frac{1}{t} + t$

References.

[Ov] A. Overbay, *Perturbative Expansion of the Colored Jones Polynomial*, University of North Carolina PhD thesis, ωεβ/Ov.

[Ro1] L. Rozansky, *A contribution of the trivial flat connection to the Jones polynomial and Witten's invariant of 3d manifolds, I*, Comm. Math. Phys. **175-2** (1996) 275–296, arXiv:hep-th/9401061.

[Ro2] L. Rozansky, *The Universal R-Matrix, Burau Representation and the Melvin-Morton Expansion of the Colored Jones Polynomial*, Adv. Math. **134-1** (1998) 1–31, arXiv:q-alg/9604005.

[Ro3] L. Rozansky, *A Universal U(1)-RCC Invariant of Links and Rationality Conjecture*, arXiv:math/0201139.



diagram	n_k^t Alexander's ω^+ / Today's / Rozansky's ρ_1^+	unknotting number / amphicheiral; genus / ribbon
	3_1^a $t-1$	1/✗
	t	1/✗
	5_1^a t^2-t+1	2/✗

	Alexander's ω^+	genus / ribbon	Rozansky's ρ_1^+	unknotting number / amphicheiral
(cut off)	… $+3t-3$	2/✗	… $+4t-4$	1/✗
7_1^a	t^3-t^2+t-1	3/✗	$3t^5+5t^3+6t$	3/✗
7_3^a	$2t^2-3t+3$	2/✗	$-9t^3+8t^2-16t+12$	2/✗
7_5^a	$2t^2-4t+5$	2/✗	$9t^3-16t^2+29t-28$	2/✗
7_7^a	t^2-5t+9	2/✗	$8-3t$	1/✗
8_2^a	$-t^3+3t^2-3t+3$	3/✗	$2t^5-8t^4+10t^3-12t^2+13t-12$	2/✗
8_4^a	$-2t^2+5t-5$	2/✗	$3t^3-8t^2+6t-4$	2/✗
8_6^a	$-2t^2+6t-7$	2/✗	$5t^3-20t^2+28t-32$	2/✗
8_8^a	$2t^2-6t+9$	2/✔	$-t^3+4t^2-12t+16$	2/✗
8_{10}^a	t^3-3t^2+6t-7	3/✗	$-t^5+4t^4-11t^3+16t^2-21t+20$	2/✗
8_{12}^a	$t^2-7t+13$	2/✗	0	2/✔
8_{14}^a	$-2t^2+8t-11$	2/✗	$5t^3-28t^2+57t-68$	1/✗
8_{16}^a	t^3-4t^2+8t-9	3/✗	$t^5-6t^4+17t^3-28t^2+35t-36$	2/✗
8_{18}^a	$-t^3+5t^2-10t+13$	3/✗	0	2/✔
8_{20}^a	t^2-2t+3	2/✔	$4t-4$	1/✗
9_1^a	$t^4-t^3+t^2-t+1$	4/✗	$4t^7+7t^5+9t^3+10t$	4/✗
9_3^a	$2t^3-3t^2+3t-3$	3/✗	$-13t^5+12t^4-25t^3+20t^2-32t+24$	3/✗
9_5^a	$6t-11$	1/✗	$100-65t$	2/✗
9_7^a	$3t^2-7t+9$	2/✗	$23t^3-56t^2+99t-108$	2/✗
9_9^a	$2t^3-4t^2+6t-7$	3/✗	$13t^5-24t^4+55t^3-72t^2+98t-96$	3/✗
9_{11}^a	$-t^3+5t^2-7t+7$	3/✗	$-2t^5+16t^4-41t^3+52t^2-66t+64$	2/✗
9_{13}^a	$4t^2-9t+11$	2/✗	$-40t^3+92t^2-154t+168$	2, 3/✗
9_{15}^a	$-2t^2+10t-15$	2/✗	$-5t^3+40t^2-108t+136$	2/✗
9_{17}^a	t^3-5t^2+9t-9	3/✗	$t^5-8t^4+23t^3-32t^2+28t-24$	2/✗
9_{19}^a	$2t^2-10t+17$	2/✗	$t^3-8t^2+20t-24$	1/✗
9_{21}^a	$-2t^2+11t-17$	2/✗	$-5t^3+44t^2-127t+164$	1/✗
9_{23}^a	$4t^2-11t+15$	2/✗	$40t^3-128t^2+243t-288$	2/✗
9_{25}^a	$-3t^2+12t-17$	2/✗	$12t^3-70t^2+153t-188$	2/✗
9_{27}^a	$-t^3+5t^2-11t+15$	3/✔	$t^3-8t^2+24t-32$	1/✗
6_3^a	t^2-3t+5	2/✗	0	1/✔
7_2^a	$3t-5$	1/✗	$14t-16$	1/✗
7_4^a	$4t-7$	1/✗	$32-24t$	2/✗
7_6^a	$-t^2+5t-7$	2/✗	$t^3-8t^2+19t-20$	1/✗
8_1^a	$7-3t$	1/✗	$5t-16$	1/✗
8_3^a	$9-4t$	1/✗	0	2/✔
8_5^a	$-t^3+3t^2-4t+5$	3/✗	$-2t^5+8t^4-13t^3+20t^2-22t+24$	2/✗
8_7^a	t^3-3t^2+5t-5	3/✗	$-t^5+4t^4-10t^3+12t^2-13t+12$	1/✗
8_9^a	$-t^3+3t^2-5t+7$	3/✔	0	1/✔
8_{11}^a	$-2t^2+7t-9$	2/✗	$5t^3-24t^2+39t-44$	1/✗
8_{13}^a	$2t^2-7t+11$	2/✗	$-t^3+4t^2-14t+20$	1/✗
8_{15}^a	$3t^2-8t+11$	2/✗	$21t^3-64t^2+120t-140$	2/✗
8_{17}^a	$-t^3+4t^2-8t+11$	3/✗	0	1/✔
8_{19}^a	t^3-t^2+1	3/✗	$-3t^5-4t^2-3t$	3/✗
8_{21}^a	$-t^2+4t-5$	2/✗	$t^3-8t^2+16t-20$	1/✗
9_2^a	$4t-7$	1/✗	$30t-40$	1/✗
9_4^a	$3t^2-5t+5$	2/✗	$23t^3-28t^2+46t-44$	2/✗
9_6^a	$2t^3-4t^2+5t-5$	3/✗	$13t^5-24t^4+45t^3-52t^2+68t-64$	3/✗
9_8^a	$-2t^2+8t-11$	2/✗	$3t^3-16t^2+29t-28$	2/✗
9_{10}^a	$4t^2-8t+9$	2/✗	$-40t^3+72t^2-114t+120$	2, 3/✗
9_{12}^a	$-2t^2+9t-13$	2/✗	$5t^3-36t^2+84t-100$	1/✗
9_{14}^a	$2t^2-9t+15$	2/✗	$-t^3+8t^2-35t+60$	1/✗
9_{16}^a	$2t^3-5t^2+8t-9$	3/✗	$-13t^5+36t^4-80t^3+120t^2-161t+168$	3/✗
9_{18}^a	$4t^2-10t+13$	2/✗	$40t^3-108t^2+193t-220$	2/✗
9_{20}^a	$-t^3+5t^2-9t+11$	3/✗	$2t^5-16t^4+47t^3-84t^2+117t-124$	2/✗
9_{22}^a	$t^3-5t^2+10t-11$	3/✗	$-t^5+8t^4-24t^3+38t^2-40t+36$	1/✗
9_{24}^a	$-t^3+5t^2-10t+13$	3/✗	$-4t^2+16t-20$	1/✗
9_{26}^a	$t^3-5t^2+11t-13$	3/✗	$-t^5+8t^4-31t^3+64t^2-85t+92$	1/✗
9_{28}^a	$t^3-5t^2+12t-15$	3/✗	$t^5-8t^4+30t^3-68t^2+105t-120$	1/✗

What else can you do with solvable approximations?: 23/26. Rolfsen table — Next: E&I Abstract, again | 1:02:21

	9^a_{15}	$-2t^2 + 10t - 15$ $-5t^3 + 40t^2 - 108t + 136$	2 / ✘ 2 / ✘
	9^a_{17}	$t^3 - 5t^2 + 9t - 9$ $t^5 - 8t^4 + 23t^3 - 32t^2 + 28t - 24$	3 / ✘ 2 / ✘
	9^a_{19}	$2t^2 - 10t + 17$ $t^3 - 8t^2 + 20t - 24$	2 / ✘ 1 / ✘
	9^a_{21}	$-2t^2 + 11t - 17$ $-5t^3 + 44t^2 - 127t + 164$	2 / ✘ 1 / ✘
	9^a_{23}	$4t^2 - 11t + 15$ $40t^3 - 128t^2 + 243t - 288$	2 / ✘ 2 / ✘
	9^a_{25}	$-3t^2 + 12t - 17$ $12t^3 - 70t^2 + 153t - 188$	2 / ✘ 2 / ✘
	9^a_{27}	$-t^3 + 5t^2 - 11t + 15$ $t^3 - 8t^2 + 24t - 32$	3 / ✔ 1 / ✘
	9^a_{29}	$t^3 - 5t^2 + 12t - 15$ $t^5 - 8t^4 + 26t^3 - 48t^2 + 59t - 56$	3 / ✘ 2 / ✘

What else can you do with solvable approximations?: 24/26. E&I Abstract, again

Next: Main Question | 1:02:47

Dror Bar-Natan: Talks: GWU-1612:

On Elves and Invariants

Follows Rozan
Overbay [Ov], j

Abstract. Whether or not you like the formulas on this page, they describe the strongest truly computable knot invariant we know.

Three steps to the computation of ρ_1:

1. **Preparation.** Given K, results

$$\langle \textit{long word} \,\|\, \textit{simple formulas} \rangle.$$

2. **Rewrite rules.** Make the *word* simpler and the *formulas* more complicated, until the *word* "*elf*" is reached

Knot K

$\downarrow$ preparation

$\langle elf \ldots elf \,\|\, \omega_0; L_0; Q_0; P_0 \rangle$

$\downarrow$ rewrite rules

$\langle elf \,\|\, \omega; -; -; P \rangle$

```
MatrixLog // PowerExpand // Simplify //
  MatrixForm
```

Enter

Question. What else can you do with solvable approximation? Chern-Simons-Witten theory is often "solved" using ideas from conformal field theory and using quantization of various moduli spaces. Does it make sense to use solvable approximation there too? Elsewhere in physics? Elsewhere in mathematics?

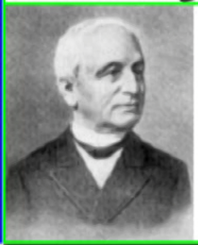
See Also. Talks at George Washington University [ωεβ/gwu], Indiana [ωεβ/ind], and Les Diablerets [ωεβ/ld], and a University of Toronto "Algebraic Knot Theory" class [ωεβ/akt].

1-Smidgen sl_2 Let $\mathfrak{g}_1$ be the 4-dimensional Lie algebra $\mathfrak{g}_1 = \langle h, e', l, f\rangle$ over the ring $R = \mathbb{Q}[\epsilon]/(\epsilon^2 = 0)$, with h central and with $[f, l] = f$, $[e', l] = -e'$, and $[e', f] = h - 2\epsilon l$. Over $\mathbb{Q}$, $\mathfrak{g}_1$

and output

$$\left\langle \ldots l_k f_k \ldots \| \, \omega; L|_{l_j \to l_k}; t^\lambda \alpha f_k + Q'; \mathbb{e}^{-q} P(\partial_\beta, \partial_\gamma) \mathbb{e}^q |_{\beta \to \alpha/\omega,\, \gamma \to \lambda \log t} \right\rangle .$$

Happy Birthday, Scott!



"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

www.katlas.org

The Knot Atlas
Anyone Can Edit

diagram	n_k^t Alexander's ω^+ / Today's / Rozansky's ρ_1^+	genus / ribbon / unknotting number / amphicheiral	dia
	5_2^a $2t-3$ $5t-4$	1 / ✘ 1 / ✘	
	6_2^a $-t^2+3t-3$ t^3-4t^2+4t-4	2 / ✘ 1 / ✘	
	7^a t^3-t^2+t-1	3 / ✘