Pensieve header: Computing ρ_1 for torus knots.

Preliminaries

In[•]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\Oaxaca-2210"];
```

In[•]:=
```
Once[<< KnotTheory`; << Rot.m];
```

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
```

```
Loading Rot.m from http://drorbn.net/la22/ap to compute rotation numbers.
```

The Program

In[•]:=
```
R1[s_, i_, j_] := s (g_{ji} (g_{j^+,j} + g_{j,j^+} - g_{ij}) - g_{ii} (g_{j,j^+} - 1) - 1/2);
Z[K_] := Z[K] = Module[{Cs, φ, n, A, s, i, j, k, Δ, G, ρ1},
    {Cs, φ} = Rot[K]; n = Length[Cs];
    A = IdentityMatrix[2n+1];
    Cases[Cs, {s_, i_, j_} :> (A[[{i, j}, {i+1, j+1}]] += ({{-T^s, T^s - 1}, {0, -1}}))];
    Δ = T^((-Total[φ]-Total[Cs[[All,1]]])/2) Det[A];
    G = Inverse[A];
    ρ1 = Sum_{k=1}^{n} R1 @@ Cs[[k]] - Sum_{k=1}^{2n} φ[[k]] (g_{kk} - 1/2);
    Factor@{Δ, Δ^2 ρ1 /. α_^+ :> α + 1 /. g_{α_,β_} :> G[[α, β]]}];
```

In[•]:=
```
T2z[p_] := Module[{q = Expand[p], n, c},
   If[q === 0, 0, c = Coefficient[q, T, n = Exponent[q, T]];
    c z^(2n) + T2z[q - c (T^(1/2) - T^(-1/2))^(2n)]]];
```

In[•]:=
```
T2z[p_] := Module[{q = Expand[p], n, c},
   If[q === 0, 0, c = Coefficient[q, T, n = Exponent[q, T]];
    c z^(2n) + T2z[q - c (T^(1/2) - T^(-1/2))^(2n)]]];
```

In[•]:=
```
ren[p_] := T^(-Exponent[p,T,Min]) p;
```

The First Few Torus Knots

The format is: Two lines for each knot. The first is a pair $\{T_{p,q}, \Delta\}$ where Δ is the Alexander polynomial. The following line is ρ_1, normalized to have no negative powers and to have its constant term non-zero.

```
In[•]:= tab = Table[ρz = Factor[ren /@ Z[K]];
    Echo@Column[{{T_{K[[1]],K[[2]]}, ρz[[1]]}, ρz[[2]]}],
    {K, TorusKnots[36]}];
```

$$\{T_{3,2},\ 1-T+T^2\}$$
$$-(-1+T)^2\left(1+T^2\right)$$

$$\{T_{5,2},\ 1-T+T^2-T^3+T^4\}$$
$$-(-1+T)^2\left(1+T^2\right)\left(2+T^2+2T^4\right)$$

$$\{T_{7,2},\ 1-T+T^2-T^3+T^4-T^5+T^6\}$$
$$-(-1+T)^2\left(1+T^2\right)\left(3+2T^2+4T^4+2T^6+3T^8\right)$$

$$\{T_{4,3},\ \left(1-T+T^2\right)\left(1-T^2+T^4\right)\}$$
$$-(-1+T)^2\left(1+T^4\right)\left(3+4T^3+3T^6\right)$$

$$\{T_{9,2},\ \left(1-T+T^2\right)\left(1-T^3+T^6\right)\}$$
$$-(-1+T)^2\left(1+T^2\right)\left(4+3T^2+6T^4+4T^6+6T^8+3T^{10}+4T^{12}\right)$$

$$\{T_{5,3},\ 1-T+T^3-T^4+T^5-T^7+T^8\}$$
$$-2(-1+T)^2(1+T)^2\left(1-T+T^2\right)\left(1-T+T^2-T^3+T^4\right)\left(2+T^3+2T^6\right)$$

$$\{T_{11,2},\ 1-T+T^2-T^3+T^4-T^5+T^6-T^7+T^8-T^9+T^{10}\}$$
$$-(-1+T)^2\left(1+T^2\right)\left(5+4T^2+8T^4+6T^6+9T^8+6T^{10}+8T^{12}+4T^{14}+5T^{16}\right)$$

$$\{T_{13,2},\ 1-T+T^2-T^3+T^4-T^5+T^6-T^7+T^8-T^9+T^{10}-T^{11}+T^{12}\}$$
$$-(-1+T)^2\left(1+T^2\right)\left(6+5T^2+10T^4+8T^6+12T^8+9T^{10}+12T^{12}+8T^{14}+10T^{16}+5T^{18}+6T^{20}\right)$$

$$\{T_{7,3},\ 1-T+T^3-T^4+T^6-T^8+T^9-T^{11}+T^{12}\}$$
$$-2(-1+T)^2(1+T)^2\left(1-T+T^2\right)\left(1-T+T^2-T^3+T^4-T^5+T^6\right)\left(3+2T^3+4T^6+2T^9+3T^{12}\right)$$

$$\{T_{5,4},\ \left(1-T+T^2-T^3+T^4\right)\left(1-T^2+T^4-T^6+T^8\right)\}$$
$$-(-1+T)^2\left(1+T^4\right)\left(2+T^4+2T^8\right)\left(3+4T^5+3T^{10}\right)$$

$$\{T_{15,2},\ \left(1-T+T^2\right)\left(1-T+T^2-T^3+T^4\right)\left(1+T-T^3-T^4-T^5+T^7+T^8\right)\}$$
$$-(-1+T)^2\left(1+T^2\right)\left(7+6T^2+12T^4+10T^6+15T^8+12T^{10}+16T^{12}+12T^{14}+15T^{16}+10T^{18}+12T^{20}+6T^{22}+7T^{24}\right)$$

$$\{T_{8,3},\ \left(1-T+T^2\right)\left(1-T^2+T^4\right)\left(1-T^4+T^8\right)\}$$
$$-(-1+T)^2\left(1+T^8\right)\left(7+12T^3+15T^6+16T^9+15T^{12}+12T^{15}+7T^{18}\right)$$

$$\{T_{17,2},\ 1-T+T^2-T^3+T^4-T^5+T^6-T^7+T^8-T^9+T^{10}-T^{11}+T^{12}-T^{13}+T^{14}-T^{15}+T^{16}\}$$
$$-(-1+T)^2\left(1+T^2\right)$$
$$\left(8+7T^2+14T^4+12T^6+18T^8+15T^{10}+20T^{12}+16T^{14}+20T^{16}+15T^{18}+18T^{20}+12T^{22}+14T^{24}+7T^{26}+8T^{28}\right)$$

$$\{T_{19,2},\ 1-T+T^2-T^3+T^4-T^5+T^6-T^7+T^8-T^9+T^{10}-T^{11}+T^{12}-T^{13}+T^{14}-T^{15}+T^{16}-T^{17}+T^{18}\}$$
$$-(-1+T)^2\left(1+T^2\right)\left(9+8T^2+16T^4+14T^6+21T^8+18T^{10}+\right.$$
$$\left.24T^{12}+20T^{14}+25T^{16}+20T^{18}+24T^{20}+18T^{22}+21T^{24}+14T^{26}+16T^{28}+8T^{30}+9T^{32}\right)$$

$$\{T_{10,3},\ \left(1-T+T^2\right)\left(1-T+T^3-T^4+T^5-T^7+T^8\right)\left(1+T-T^3-T^4-T^5+T^7+T^8\right)\}$$
$$-(-1+T)^2\left(1+T^2\right)\left(1-T^2+T^4-T^6+T^8\right)\left(9+16T^3+21T^6+24T^9+25T^{12}+24T^{15}+21T^{18}+16T^{21}+9T^{24}\right)$$

$$\{T_{7,4},\ \left(1-T+T^2-T^3+T^4-T^5+T^6\right)\left(1-T^2+T^4-T^6+T^8-T^{10}+T^{12}\right)\}$$
$$-(-1+T)^2\left(1+T^4\right)\left(3+4T^7+3T^{14}\right)\left(3+2T^4+4T^8+2T^{12}+3T^{16}\right)$$

$$\left\{T_{21,2},\ \left(1-T+T^2\right)\ \left(1-T+T^2-T^3+T^4-T^5+T^6\right)\ \left(1+T-T^3-T^4+T^6-T^8-T^9+T^{11}+T^{12}\right)\right\}$$

» $$-(-1+T)^2\ \left(1+T^2\right)\ \left(10+9\,T^2+18\,T^4+16\,T^6+24\,T^8+21\,T^{10}+28\,T^{12}+24\,T^{14}+30\,T^{16}+25\,T^{18}+30\,T^{20}+24\,T^{22}+28\,T^{24}+21\,T^{26}+24\,T^{28}+16\,T^{30}+18\,T^{32}+9\,T^{34}+10\,T^{36}\right)$$

$$\left\{T_{11,3},\ 1-T+T^3-T^4+T^6-T^7+T^9-T^{10}+T^{11}-T^{13}+T^{14}-T^{16}+T^{17}-T^{19}+T^{20}\right\}$$

» $$-2\ (-1+T)^2\ (1+T)^2\ \left(1-T+T^2\right)\ \left(1-T+T^2-T^3+T^4-T^5+T^6-T^7+T^8-T^9+T^{10}\right)\ \left(5+4\,T^3+8\,T^6+6\,T^9+9\,T^{12}+6\,T^{15}+8\,T^{18}+4\,T^{21}+5\,T^{24}\right)$$

$$\left\{T_{23,2},\ 1-T+T^2-T^3+T^4-T^5+T^6-T^7+T^8-T^9+T^{10}-T^{11}+T^{12}-T^{13}+T^{14}-T^{15}+T^{16}-T^{17}+T^{18}-T^{19}+T^{20}-T^{21}+T^{22}\right\}$$

» $$-(-1+T)^2\ \left(1+T^2\right)\ \left(11+10\,T^2+20\,T^4+18\,T^6+27\,T^8+24\,T^{10}+32\,T^{12}+28\,T^{14}+35\,T^{16}+30\,T^{18}+36\,T^{20}+30\,T^{22}+35\,T^{24}+28\,T^{26}+32\,T^{28}+24\,T^{30}+27\,T^{32}+18\,T^{34}+20\,T^{36}+10\,T^{38}+11\,T^{40}\right)$$

» $$\left\{T_{6,5},\ \left(1-T+T^2-T^3+T^4\right)\ \left(1-T+T^3-T^4+T^5-T^7+T^8\right)\ \left(1+T-T^3-T^4-T^5+T^7+T^8\right)\right\}$$
$$-(-1+T)^2\ \left(1+T^2\right)\ \left(1-T^2+T^4\right)\ \left(2+T^6+2\,T^{12}\right)\ \left(5+8\,T^5+9\,T^{10}+8\,T^{15}+5\,T^{20}\right)$$

$$\left\{T_{25,2},\ \left(1-T+T^2-T^3+T^4\right)\ \left(1-T^5+T^{10}-T^{15}+T^{20}\right)\right\}$$

» $$-(-1+T)^2\ \left(1+T^2\right)\ \left(12+11\,T^2+22\,T^4+20\,T^6+30\,T^8+27\,T^{10}+36\,T^{12}+32\,T^{14}+40\,T^{16}+35\,T^{18}+42\,T^{20}+36\,T^{22}+42\,T^{24}+35\,T^{26}+40\,T^{28}+32\,T^{30}+36\,T^{32}+27\,T^{34}+30\,T^{36}+20\,T^{38}+22\,T^{40}+11\,T^{42}+12\,T^{44}\right)$$

$$\left\{T_{13,3},\ 1-T+T^3-T^4+T^6-T^7+T^9-T^{10}+T^{12}-T^{14}+T^{15}-T^{17}+T^{18}-T^{20}+T^{21}-T^{23}+T^{24}\right\}$$

» $$-2\ (-1+T)^2\ (1+T)^2\ \left(1-T+T^2\right)\ \left(1-T+T^2-T^3+T^4-T^5+T^6-T^7+T^8-T^9+T^{10}-T^{11}+T^{12}\right)\ \left(6+5\,T^3+10\,T^6+8\,T^9+12\,T^{12}+9\,T^{15}+12\,T^{18}+8\,T^{21}+10\,T^{24}+5\,T^{27}+6\,T^{30}\right)$$

» $$\left\{T_{9,4},\ \left(1-T+T^2\right)\ \left(1-T^2+T^4\right)\ \left(1-T^3+T^6\right)\ \left(1-T^6+T^{12}\right)\right\}$$
$$-(-1+T)^2\ \left(1+T^4\right)\ \left(3+4\,T^9+3\,T^{18}\right)\ \left(4+3\,T^4+6\,T^8+4\,T^{12}+6\,T^{16}+3\,T^{20}+4\,T^{24}\right)$$

$$\left\{T_{27,2},\ \left(1-T+T^2\right)\ \left(1-T^3+T^6\right)\ \left(1-T^9+T^{18}\right)\right\}$$

» $$-(-1+T)^2\ \left(1+T^2\right)\ \left(13+12\,T^2+24\,T^4+22\,T^6+33\,T^8+30\,T^{10}+40\,T^{12}+36\,T^{14}+45\,T^{16}+40\,T^{18}+48\,T^{20}+42\,T^{22}+49\,T^{24}+42\,T^{26}+48\,T^{28}+40\,T^{30}+45\,T^{32}+36\,T^{34}+40\,T^{36}+30\,T^{38}+33\,T^{40}+22\,T^{42}+24\,T^{44}+12\,T^{46}+13\,T^{48}\right)$$

$$\left\{T_{7,5},\ 1-T+T^5-T^6+T^7-T^8+T^{10}-T^{11}+T^{12}-T^{13}+T^{14}-T^{16}+T^{17}-T^{18}+T^{19}-T^{23}+T^{24}\right\}$$

» $$-2\ (-1+T)^2\ (1+T)^2\ \left(1-T+T^2-T^3+T^4\right)\ \left(1-T+T^2-T^3+T^4-T^5+T^6\right)\ \left(2+T^7+2\,T^{14}\right)\ \left(3+2\,T^5+4\,T^{10}+2\,T^{15}+3\,T^{20}\right)$$

$$\left\{T_{14,3},\ \left(1-T+T^2\right)\ \left(1-T+T^3-T^4+T^6-T^8+T^9-T^{11}+T^{12}\right)\ \left(1+T-T^3-T^4+T^6-T^8-T^9+T^{11}+T^{12}\right)\right\}$$

» $$-(-1+T)^2\ \left(1+T^2\right)\ \left(1-T^2+T^4-T^6+T^8-T^{10}+T^{12}\right)\ \left(13+24\,T^3+33\,T^6+40\,T^9+45\,T^{12}+48\,T^{15}+49\,T^{18}+48\,T^{21}+45\,T^{24}+40\,T^{27}+33\,T^{30}+24\,T^{33}+13\,T^{36}\right)$$

$$\left\{T_{29,2},\ 1-T+T^2-T^3+T^4-T^5+T^6-T^7+T^8-T^9+T^{10}-T^{11}+T^{12}-T^{13}+T^{14}-T^{15}+T^{16}-T^{17}+T^{18}-T^{19}+T^{20}-T^{21}+T^{22}-T^{23}+T^{24}-T^{25}+T^{26}-T^{27}+T^{28}\right\}$$

» $$-(-1+T)^2\ \left(1+T^2\right)\ \left(14+13\,T^2+26\,T^4+24\,T^6+36\,T^8+33\,T^{10}+44\,T^{12}+40\,T^{14}+50\,T^{16}+45\,T^{18}+54\,T^{20}+48\,T^{22}+56\,T^{24}+49\,T^{26}+56\,T^{28}+48\,T^{30}+54\,T^{32}+45\,T^{34}+50\,T^{36}+40\,T^{38}+44\,T^{40}+33\,T^{42}+36\,T^{44}+24\,T^{46}+26\,T^{48}+13\,T^{50}+14\,T^{52}\right)$$

$$\left\{T_{31,2},\ 1-T+T^2-T^3+T^4-T^5+T^6-T^7+T^8-T^9+T^{10}-T^{11}+T^{12}-T^{13}+T^{14}-T^{15}+T^{16}-T^{17}+T^{18}-T^{19}+T^{20}-T^{21}+T^{22}-T^{23}+T^{24}-T^{25}+T^{26}-T^{27}+T^{28}-T^{29}+T^{30}\right\}$$

» $$-(-1+T)^2\ \left(1+T^2\right)\ \left(15+14\,T^2+28\,T^4+26\,T^6+39\,T^8+36\,T^{10}+48\,T^{12}+44\,T^{14}+55\,T^{16}+50\,T^{18}+60\,T^{20}+54\,T^{22}+63\,T^{24}+56\,T^{26}+64\,T^{28}+56\,T^{30}+63\,T^{32}+54\,T^{34}+60\,T^{36}+50\,T^{38}+55\,T^{40}+44\,T^{42}+48\,T^{44}+36\,T^{46}+39\,T^{48}+26\,T^{50}+28\,T^{52}+14\,T^{54}+15\,T^{56}\right)$$

$$\{T_{8,5},\ (1-T+T^2-T^3+T^4)\ (1-T^2+T^4-T^6+T^8)\ (1-T^4+T^8-T^{12}+T^{16})\}$$
$$-(-1+T)^2\ (1+T^8)\ (2+T^8+2\,T^{16})\ (7+12\,T^5+15\,T^{10}+16\,T^{15}+15\,T^{20}+12\,T^{25}+7\,T^{30})$$

$$\{T_{16,3},\ (1-T+T^2)\ (1-T^2+T^4)\ (1-T^4+T^8)\ (1-T^8+T^{16})\}$$
$$-(-1+T)^2\ (1+T^{16})\ (15+28\,T^3+39\,T^6+48\,T^9+55\,T^{12}+60\,T^{15}+63\,T^{18}+64\,T^{21}+63\,T^{24}+60\,T^{27}+55\,T^{30}+48\,T^{33}+39\,T^{36}+28\,T^{39}+15\,T^{42})$$

$$\{T_{11,4},\ (1-T+T^2-T^3+T^4-T^5+T^6-T^7+T^8-T^9+T^{10})\ (1-T^2+T^4-T^6+T^8-T^{10}+T^{12}-T^{14}+T^{16}-T^{18}+T^{20})\}$$
$$-(-1+T)^2\ (1+T^4)\ (3+4\,T^{11}+3\,T^{22})\ (5+4\,T^4+8\,T^8+6\,T^{12}+9\,T^{16}+6\,T^{20}+8\,T^{24}+4\,T^{28}+5\,T^{32})$$

$$\{T_{33,2},\ (1-T+T^2)\ (1-T+T^2-T^3+T^4-T^5+T^6-T^7+T^8-T^9+T^{10})\ (1+T-T^3-T^4+T^6+T^7-T^9-T^{10}-T^{11}+T^{13}+T^{14}-T^{16}-T^{17}+T^{19}+T^{20})\}$$
$$-(-1+T)^2\ (1+T^2)\ (16+15\,T^2+30\,T^4+28\,T^6+42\,T^8+39\,T^{10}+52\,T^{12}+48\,T^{14}+60\,T^{16}+55\,T^{18}+66\,T^{20}+60\,T^{22}+70\,T^{24}+63\,T^{26}+72\,T^{28}+64\,T^{30}+72\,T^{32}+63\,T^{34}+70\,T^{36}+60\,T^{38}+66\,T^{40}+55\,T^{42}+60\,T^{44}+48\,T^{46}+52\,T^{48}+39\,T^{50}+42\,T^{52}+28\,T^{54}+30\,T^{56}+15\,T^{58}+16\,T^{60})$$

$$\{T_{17,3},\ 1-T+T^3-T^4+T^6-T^7+T^9-T^{10}+T^{12}-T^{13}+T^{15}-T^{16}+T^{17}-T^{19}+T^{20}-T^{22}+T^{23}-T^{25}+T^{26}-T^{28}+T^{29}-T^{31}+T^{32}\}$$
$$-2\ (-1+T)^2\ (1+T)^2\ (1-T+T^2)\ (1-T+T^2-T^3+T^4-T^5+T^6-T^7+T^8-T^9+T^{10}-T^{11}+T^{12}-T^{13}+T^{14}-T^{15}+T^{16})\ (8+7\,T^3+14\,T^6+12\,T^9+18\,T^{12}+15\,T^{15}+20\,T^{18}+16\,T^{21}+20\,T^{24}+15\,T^{27}+18\,T^{30}+12\,T^{33}+14\,T^{36}+7\,T^{39}+8\,T^{42})$$

$$\{T_{7,6},\ (1-T+T^2-T^3+T^4-T^5+T^6)\ (1-T+T^3-T^4+T^6-T^8+T^9-T^{11}+T^{12})\ (1+T-T^3-T^4+T^6-T^8-T^9+T^{11}+T^{12})\}$$
$$-(-1+T)^2\ (1+T^2)\ (1-T^2+T^4)\ (3+2\,T^6+4\,T^{12}+2\,T^{18}+3\,T^{24})\ (5+8\,T^7+9\,T^{14}+8\,T^{21}+5\,T^{28})$$

$$\{T_{35,2},\ (1-T+T^2-T^3+T^4)\ (1-T+T^2-T^3+T^4-T^5+T^6)\ (1+T-T^5-T^6-T^7-T^8+T^{10}+T^{11}+T^{12}+T^{13}+T^{14}-T^{16}-T^{17}-T^{18}-T^{19}+T^{23}+T^{24})\}$$
$$-(-1+T)^2\ (1+T^2)\ (17+16\,T^2+32\,T^4+30\,T^6+45\,T^8+42\,T^{10}+56\,T^{12}+52\,T^{14}+65\,T^{16}+60\,T^{18}+72\,T^{20}+66\,T^{22}+77\,T^{24}+70\,T^{26}+80\,T^{28}+72\,T^{30}+81\,T^{32}+72\,T^{34}+80\,T^{36}+70\,T^{38}+77\,T^{40}+66\,T^{42}+72\,T^{44}+60\,T^{46}+65\,T^{48}+52\,T^{50}+56\,T^{52}+42\,T^{54}+45\,T^{56}+30\,T^{58}+32\,T^{60}+16\,T^{62}+17\,T^{64})$$

$$\{T_{9,5},\ (1-T+T^3-T^4+T^5-T^7+T^8)\ (1-T^3+T^9-T^{12}+T^{15}-T^{21}+T^{24})\}$$
$$-2\ (-1+T)^2\ (1+T)^2\ (1-T+T^2)\ (1-T+T^2-T^3+T^4)\ (1-T^3+T^6)\ (2+T^9+2\,T^{18})\ (4+3\,T^5+6\,T^{10}+4\,T^{15}+6\,T^{20}+3\,T^{25}+4\,T^{30})$$