

Pensieve header: Developing ρ_d .

Program

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\Oaxaca-2210"];
```

(Alt) In[]:=

```
Once[<< KnotTheory` ; << Rot.m];
```

Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.

Read more at <http://katlas.org/wiki/KnotTheory>.

Loading Rot.m from <http://drorbn.net/la22/ap> to compute rotation numbers.

(Alt) In[]:=

```
<< "../../Projects/Profile/Profile.m"
```

This is Profile.m of <http://www.drorbn.net/AcademicPensieve/Projects/Profile/>.

This version: April 2020. Original version: July 1994.

(Alt) In[]:=

```
{p*, x*, pi*, xi*, pbar, xbar, pi_bar, xi_bar} = {pi, xi, p, x, pi_bar, xi_bar, pbar, xbar}; (u_{i_})^* := (u^*)_i;
```

(Alt) In[]:=

```
Zip_{i}[E_] := E;
Zip_{E, S}[E_] := (Collect[E // Zip_{S}, S] /. f_ . S^d_ -> (D[f, {S*, d}])) /. S* -> 0
```

(Alt) In[]:=

```
{ca_{1,2} = 1, ca_{1,10} = -1, ca_{2,1} = 0, cb_{2,10} = 3/2};
```

(Alt) In[]:=

```
V@gamma_{d,0}[j_] := 0; V@gamma_{1,0}[k_] := phi ( (ca_{1,2}/2 + p_k x_k ca_{1,10}) /. v : p | x -> vbar;
```

(Alt) In[]:=

```
V@gamma_{2,1}[k_] := -ca_{2,1} + p_k x_k (-ca_{1,2} ca_{1,10} - cb_{2,10}) /. v : p | x -> vbar;
V@gamma_{2,-1}[k_] := ca_{2,1} + p_k x_k (ca_{1,2} ca_{1,10} - ca_{1,10}^2 + cb_{2,10}) /. v : p | x -> vbar;
```

(Alt) In[]:=

```
V@r_{1,1}[i_, j_] := -ca_{1,2}/2 + p_i x_i ca_{1,2} - p_j x_j ca_{1,2} -
1/2 (-1 + T) p_i p_j x_i^2 ca_{1,10} + 1/2 (-1 + T) p_j^2 x_j^2 ca_{1,10} + p_i p_j x_i x_j ca_{1,10} - p_j^2 x_j x_j ca_{1,10};
```

(Alt) In[]:=

$$\begin{aligned}
 \mathbf{V@r}_{1,-1}[i_-, j_-] := & \frac{ca_{1,2}}{2} - p_i x_i ca_{1,2} + p_j x_i ca_{1,2} - \\
 & \frac{(-1+T) p_i p_j x_i^2 ca_{1,10}}{2T} + \frac{(-1+T) p_j^2 x_i^2 ca_{1,10}}{2T} - p_i p_j x_i x_j ca_{1,10} + p_j^2 x_i x_j ca_{1,10};
 \end{aligned}$$

(Alt) In[]:=

$$\begin{aligned}
 \mathbf{V@r}_{2,1}[i_-, j_-] := & \frac{1}{3} (-1+T) p_i^2 p_j x_i^3 ca_{1,10}^2 - \frac{1}{6} (-1+T) (5+T) p_i p_j^2 x_i^3 ca_{1,10}^2 + \frac{1}{6} (-1+T) (3+T) p_j^3 x_i^3 ca_{1,10}^2 - \\
 & \frac{1}{2} p_i^2 p_j x_i^2 x_j ca_{1,10}^2 + \frac{1}{2} (2+T) p_i p_j^2 x_i^2 x_j ca_{1,10}^2 - \frac{1}{2} (1+T) p_j^3 x_i^2 x_j ca_{1,10}^2 - \frac{1}{2} p_i p_j^2 x_i x_j^2 ca_{1,10}^2 + \\
 & \frac{1}{2} p_j^3 x_i x_j^2 ca_{1,10}^2 + \frac{1}{2} p_i x_i (-ca_{1,2}^2 - 4 ca_{2,1}) + ca_{2,1} + \frac{1}{2} p_j x_i (ca_{1,2}^2 + 4 ca_{2,1}) + \\
 & p_i p_j x_i x_j (-2 ca_{1,2} ca_{1,10} + ca_{1,10}^2 - cb_{2,10}) + p_j^2 x_i x_j (2 ca_{1,2} ca_{1,10} - ca_{1,10}^2 + cb_{2,10}) + \\
 & \frac{1}{2} p_j^2 x_i^2 (ca_{1,2} ca_{1,10} - 2T ca_{1,2} ca_{1,10} - ca_{1,10}^2 + T ca_{1,10}^2 + cb_{2,10} - T cb_{2,10}) + \\
 & \frac{1}{2} p_i p_j x_i^2 (-ca_{1,2} ca_{1,10} + 2T ca_{1,2} ca_{1,10} + ca_{1,10}^2 - T ca_{1,10}^2 - cb_{2,10} + T cb_{2,10});
 \end{aligned}$$

(Alt) In[]:=

$$\begin{aligned}
 \mathbf{V@r}_{2,-1}[i_-, j_-] := & -\frac{(-1+T) p_i^2 p_j x_i^3 ca_{1,10}^2}{3T} + \frac{(-1+T) (1+5T) p_i p_j^2 x_i^3 ca_{1,10}^2}{6T^2} - \\
 & \frac{(-1+T) (1+3T) p_j^3 x_i^3 ca_{1,10}^2}{6T^2} - \frac{1}{2} p_i^2 p_j x_i^2 x_j ca_{1,10}^2 + \frac{(1+2T) p_i p_j^2 x_i^2 x_j ca_{1,10}^2}{2T} - \\
 & \frac{(1+T) p_j^3 x_i^2 x_j ca_{1,10}^2}{2T} - \frac{1}{2} p_i p_j^2 x_i x_j^2 ca_{1,10}^2 + \frac{1}{2} p_j^3 x_i x_j^2 ca_{1,10}^2 + \frac{1}{2} p_j x_i (ca_{1,2}^2 - 4 ca_{2,1}) - \\
 & ca_{2,1} + \frac{1}{2} p_i x_i (-ca_{1,2}^2 + 4 ca_{2,1}) + p_i p_j x_i x_j cb_{2,10} - p_j^2 x_i x_j cb_{2,10} + \\
 & \frac{p_i p_j x_i^2 (T ca_{1,2} ca_{1,10} - cb_{2,10} + T cb_{2,10})}{2T} - \frac{p_j^2 x_i^2 (T ca_{1,2} ca_{1,10} - cb_{2,10} + T cb_{2,10})}{2T};
 \end{aligned}$$

(Alt) In[]:=

```

gPair[1] = 1;
gPair[ed·Bs_] := ed gPair[Bs];
gPair[c_?NumberQ * Bs_] := c gPair[Bs];
gPair[ε_Plus] := gPair /@ ε;
gPair[rd,s[i_, j_] p] := gPair[{rd,s[i, j]p};
gPair[γd,φ[k_] p] := gPair[{γd,φ[k]p};
gPair[Bs_Times] := gPair[List@@Bs];
gPair[Bs_List] := Module[{es, BBs},
  BBs = Bs /. e_ p -> Sequence @@ Table[e, {p}];
  es = Union @@ (List @@@ BBs);
  Do[v@i = es[[i]], {i, Length@es}];
  gpair[Replace[BBs, Thread[es -> Range@Length@es], {2}]] /. gα,β -> gv@α, v@β
]

```

(Alt) In[]:=

```

gpair[Bs_List] := gpair[Bs] = Module[{es},
  Print["At gpair with Bs=", Bs];
  es = Union @@ (List @@@ Bs);
  Simplify@ZipJoin@@Table[{pα, p̄α, xα, x̄α}, {α, es}] [Times [
    Times @@ (Bs /. {
      rd,s[i_, j_] -> V[rd,s[i, j]], γd,φ[k_] -> V[γd,φ[k]]
    }),
    Exp[Sum[gα,β (πα + π̄α) (ξβ + ξ̄β), {α, es}, {β, es}] - Sum[ξ̄α πα, {α, es}]]
]]
]

```

(Alt) In[]:=

```

ρd_ [K_] := ρd [K] = PPρd@Module [ {Cs, φ, n, A, s, i, j, k, Δ, G, d1, ρd1, ρd2, ρd3, ρd4},
  PPGreen [
    {Cs, φ} = Rot [K]; n = Length [Cs];
    A = IdentityMatrix [2 n + 1];
    Cases [Cs, {s_, i_, j_} => (A[[{i, j}, {i + 1, j + 1}]] += ( -Ts Ts - 1 ))];
    G = Inverse [A];
  ];
  ρd1 = PPMold@Exp [Total [Cases [Cs, {s_, i_, j_} => Sum [ed1 rd1,s [i, j], {d1, d}]]] +
    Sum [ed1 γd1,φ[[k]] [k], {k, 2 n}, {d1, d}]];
  ρd2 = PPExpandedMold@Table [Expand@SeriesCoefficient [ρd1, {ε, 0, d1}], {d1, d}];
  ρd3 = PPPairing [gPair /@ ρd2];
  ρd4 = PPSubstitution@Factor [Join [{Δ}, ρd3 * Δ2 Range [d]] /.
    {gα,β => G [[α, β]], Δ → T(-Total [φ] - Total [Cs [[All, 1]]) / 2 Det [A]}]
];

```

Testing

(Alt) In[]:=

gPair [r_{1,-1} [3, 4]]

At gpair with Bs = {r_{1,-1} [1, 2]}

(Alt) Out[]:=

$$\frac{1}{2} + \left(-1 + \frac{1}{T}\right) g_{4,3}^2 + g_{4,3} (1 + g_{3,4} - 2 g_{4,4}) + g_{3,3} \left(-1 + \frac{(-1 + T) g_{4,3}}{T} + g_{4,4}\right)$$

(Alt) In[]:=

gPair [r_{1,-1} [5, 6]]

(Alt) Out[]:=

$$\frac{1}{2} + \left(-1 + \frac{1}{T}\right) g_{6,5}^2 + g_{6,5} (1 + g_{5,6} - 2 g_{6,6}) + g_{5,5} \left(-1 + \frac{(-1 + T) g_{6,5}}{T} + g_{6,6}\right)$$

(Alt) In[]:=

gPair [γ_{1,-1} [3]]

At gpair with Bs = {γ_{1,-1} [1]}

(Alt) Out[]:=

$$-\frac{1}{2} + g_{3,3}$$

(Alt) In[]:=

gPair [γ_{1,0} [6]]

At gpair with Bs = {γ_{1,0} [1]}

(Alt) Out[]:=

$$0$$

(Alt) In[]:=

gPair[$\gamma_{1,-1}[3]^2$]

At gpair with Bs={ $\gamma_{1,-1}[1]$, $\gamma_{1,-1}[1]$ }

(Alt) Out[]:=

$$\frac{1}{4} - g_{3,3} + 2 g_{3,3}^2$$

(Alt) In[]:=

gPair[$r_{2,1}[3, 4]$ $\gamma_{1,-1}[3]$]

At gpair with Bs={ $r_{2,1}[1, 2]$, $\gamma_{1,-1}[1]$ }

(Alt) Out[]:=

$$\begin{aligned} & \frac{1}{4} \left(4 g_{3,3}^3 \left(8 (-1 + T) g_{4,3} - 3 g_{4,4} \right) - g_{4,3} \left(1 + 8 g_{3,4}^2 g_{4,3} + 2 (-3 + 2 T + T^2) g_{4,3}^2 + \right. \right. \\ & \quad \left. \left. 3 g_{3,4} (1 + 2 g_{4,3}) (3 + 2 (1 + T) g_{4,3} - 4 g_{4,4}) - 6 g_{4,4} + 6 g_{4,4}^2 - g_{4,3} (1 - 3 T + 6 (1 + T) g_{4,4}) \right) - \right. \\ & \quad \left. 2 g_{3,3}^2 \left(2 + 8 (-5 + 4 T + T^2) g_{4,3}^2 - 11 g_{4,4} + 4 g_{4,4}^2 + g_{4,3} (-13 + 19 T + 18 g_{3,4} - 12 (2 + T) g_{4,4}) \right) + \right. \\ & \quad \left. g_{3,3} \left(3 + 16 (-3 + 2 T + T^2) g_{4,3}^3 - 9 g_{4,4} + 6 g_{4,4}^2 + 6 g_{4,3}^2 (-6 + 7 T + T^2 + 4 (2 + T) g_{3,4} - 6 (1 + T) g_{4,4}) + \right. \right. \\ & \quad \left. \left. g_{4,3} (1 + 9 T + g_{3,4} (44 - 32 g_{4,4}) - 12 (4 + T) g_{4,4} + 24 g_{4,4}^2) \right) \right) \end{aligned}$$

(Alt) In[]:=

ρ_1 [Knot[3, 1]]

KnotTheory: Loading precomputed data in PD4Knots`.

At gpair with Bs={ $r_{1,-1}[2, 1]$ }

(Alt) Out[]:=

$$\left\{ \frac{1 - T + T^2}{T}, \frac{(-1 + T)^2 (1 + T^2)}{T^2} \right\}$$

(Alt) In[]:=

TableForm[**Table**[**Join**[{**K[[1]]**_{K[[2]]}}, ρ_1 [**K**]], {**K**, **AllKnots**[{3, 6}]}], **TableAlignments** \rightarrow **Center**]

At gpair with Bs={ $r_{1,1}[1, 2]$ }

At gpair with Bs={ $\gamma_{1,1}[1]$ }

At gpair with Bs={ $r_{1,1}[2, 1]$ }

(Alt) Out[]//TableForm=

3_1	$\frac{1-T+T^2}{T}$	$\frac{(-1+T)^2 (1+T^2)}{T^2}$
4_1	$-\frac{1-3T+T^2}{T}$	0
5_1	$\frac{1-T+T^2-T^3+T^4}{T^2}$	$\frac{(-1+T)^2 (1+T^2) (2+T^2+2T^4)}{T^4}$
5_2	$\frac{2-3T+2T^2}{T}$	$\frac{(-1+T)^2 (5-4T+5T^2)}{T^2}$
6_1	$-\frac{(-2+T) (-1+2T)}{T}$	$\frac{(-1+T)^2 (1-4T+T^2)}{T^2}$
6_2	$-\frac{1-3T+3T^2-3T^3+T^4}{T^2}$	$\frac{(-1+T)^2 (1-4T+4T^2-4T^3+4T^4-4T^5+T^6)}{T^4}$
6_3	$\frac{1-3T+5T^2-3T^3+T^4}{T^2}$	0

(Alt) In[]:=

$ca_{1,10} = -ca_{1,2}$

(Alt) In[]:=

TableForm[Table[Join[{K[[1]]_{K[[2]]}}, ρ₁[K]], {K, AllKnots[{3, 6}]}], TableAlignments → Center]

(Alt) Out[]//TableForm=

3₁	$\frac{1-T+T^2}{T}$	$\frac{(-1+T)^2 (1+T^2)}{T^2}$
4₁	$-\frac{1-3T+T^2}{T}$	0
5₁	$\frac{1-T+T^2-T^3+T^4}{T^2}$	$\frac{(-1+T)^2 (1+T^2) (2+T^2+2T^4)}{T^4}$
5₂	$\frac{2-3T+2T^2}{T}$	$\frac{(-1+T)^2 (5-4T+5T^2)}{T^2}$
6₁	$-\frac{(-2+T) (-1+2T)}{T}$	$\frac{(-1+T)^2 (1-4T+T^2)}{T^2}$
6₂	$-\frac{1-3T+3T^2-3T^3+T^4}{T^2}$	$\frac{(-1+T)^2 (1-4T+4T^2-4T^3+4T^4-4T^5+T^6)}{T^4}$
6₃	$\frac{1-3T+5T^2-3T^3+T^4}{T^2}$	0

(Alt) In[]:=

ca_{1,2} = 1;

(Alt) In[]:=

TableForm[Table[Join[{K[[1]]_{K[[2]]}}, ρ₁[K]], {K, AllKnots[{3, 6}]}], TableAlignments → Center]

(Alt) Out[]//TableForm=

3₁	$\frac{1-T+T^2}{T}$	$\frac{(-1+T)^2 (1+T^2)}{T^2}$
4₁	$-\frac{1-3T+T^2}{T}$	0
5₁	$\frac{1-T+T^2-T^3+T^4}{T^2}$	$\frac{(-1+T)^2 (1+T^2) (2+T^2+2T^4)}{T^4}$
5₂	$\frac{2-3T+2T^2}{T}$	$\frac{(-1+T)^2 (5-4T+5T^2)}{T^2}$
6₁	$-\frac{(-2+T) (-1+2T)}{T}$	$\frac{(-1+T)^2 (1-4T+T^2)}{T^2}$
6₂	$-\frac{1-3T+3T^2-3T^3+T^4}{T^2}$	$\frac{(-1+T)^2 (1-4T+4T^2-4T^3+4T^4-4T^5+T^6)}{T^4}$
6₃	$\frac{1-3T+5T^2-3T^3+T^4}{T^2}$	0

(Alt) In[]:=

z1 = Factor@ρ₂[Knot[3, 1]]

At gpair with Bs={r_{1,-1}[1, 2], r_{1,-1}[1, 2]}

At gpair with Bs={r_{1,-1}[2, 4], r_{1,-1}[3, 1]}

At gpair with Bs={r_{1,-1}[2, 1], r_{1,-1}[2, 1]}

At gpair with Bs={r_{1,-1}[1, 3], r_{1,-1}[4, 2]}

At gpair with Bs={r_{1,-1}[3, 1], r_{1,-1}[4, 2]}

At gpair with Bs={r_{2,-1}[1, 2]}

At gpair with Bs={r_{2,-1}[2, 1]}

At gpair with Bs={r_{1,-1}[1, 3], γ_{1,-1}[2]}

At gpair with Bs={r_{1,-1}[2, 1], γ_{1,-1}[2]}

At gpair with Bs={r_{1,-1}[3, 1], γ_{1,-1}[2]}

At gpair with Bs={r_{1,-1}[2, 3], γ_{1,0}[1]}

At gpair with Bs={r_{1,-1}[2, 1], γ_{1,0}[1]}

At gpair with Bs={r_{1,-1}[3, 2], γ_{1,0}[1]}

At gpair with Bs={γ_{1,-1}[2], γ_{1,0}[1]}

At gpair with Bs={γ_{1,0}[1], γ_{1,0}[1]}

At gpair with Bs={r_{1,-1}[1, 2], γ_{1,0}[1]}

At gpair with Bs={r_{1,-1}[3, 1], γ_{1,0}[2]}

At gpair with Bs={γ_{1,0}[1], γ_{1,0}[2]}

At gpair with Bs={r_{1,-1}[1, 3], γ_{1,0}[2]}

At gpair with Bs={r_{1,-1}[1, 2], γ_{1,0}[2]}

At gpair with Bs={r_{1,-1}[2, 1], γ_{1,0}[3]}

At gpair with Bs={γ_{1,-1}[1], γ_{1,0}[2]}

At gpair with Bs={r_{1,-1}[1, 2], γ_{1,0}[3]}

At gpair with Bs={r_{1,-1}[2, 1], γ_{1,0}[2]}

At gpair with Bs={γ_{2,-1}[1]}

At gpair with Bs={γ_{2,0}[1]}

(Alt) Out[]:=

$$\left\{ \frac{1 - T + T^2}{T}, \frac{(-1 + T)^2 (1 + T^2)}{T^2}, \frac{1 - 4 T + 7 T^2 - 12 T^3 + 18 T^4 - 12 T^5 + 7 T^6 - 4 T^7 + T^8}{2 T^4} \right\}$$

(Alt) In[]:=

PolynomialRemainder [

$$\frac{1}{2 T^4} (1 - T - 5 T^2 + 18 T^3 - 30 T^4 + 42 T^5 - 35 T^6 + 17 T^7 - 5 T^8 - 4 ca_{2,1} + 12 T ca_{2,1} - 20 T^2 ca_{2,1} + 16 T^3 ca_{2,1} - 16 T^5 ca_{2,1} + 20 T^6 ca_{2,1} - 12 T^7 ca_{2,1} + 4 T^8 ca_{2,1} - 2 T cb_{2,10} + 8 T^2 cb_{2,10} - 20 T^3 cb_{2,10} + 32 T^4 cb_{2,10} - 36 T^5 cb_{2,10} + 28 T^6 cb_{2,10} - 14 T^7 cb_{2,10} + 4 T^8 cb_{2,10}), 1 - T + T^2, T]$$

(Alt) Out[]:=

3

(Alt) In[]:=

Simplify[z1[[3]] - (z1[[3]] /. T → T⁻¹)]

(Alt) Out[]:=

0

(Alt) In[]:=

z2 = Factor@ρ₂[Knot[4, 1]]At gpair with Bs={r_{1,-1}[2, 3], r_{1,-1}[4, 1]}At gpair with Bs={r_{1,-1}[2, 4], r_{1,1}[1, 3]}At gpair with Bs={r_{1,-1}[4, 2], r_{1,1}[1, 3]}At gpair with Bs={r_{1,1}[1, 2], r_{1,1}[1, 2]}At gpair with Bs={r_{1,-1}[1, 3], r_{1,1}[2, 4]}At gpair with Bs={r_{1,-1}[3, 1], r_{1,1}[2, 4]}At gpair with Bs={r_{1,1}[1, 2], r_{1,1}[3, 4]}At gpair with Bs={r_{2,1}[1, 2]}At gpair with Bs={r_{1,1}[1, 2], γ_{1,-1}[2]}At gpair with Bs={r_{1,1}[2, 3], γ_{1,-1}[1]}At gpair with Bs={r_{1,-1}[1, 2], γ_{1,-1}[3]}At gpair with Bs={r_{1,1}[1, 2], γ_{1,-1}[3]}At gpair with Bs={r_{1,1}[1, 3], γ_{1,-1}[2]}At gpair with Bs={γ_{1,-1}[1], γ_{1,-1}[2]}At gpair with Bs={r_{1,1}[1, 2], γ_{1,0}[1]}At gpair with Bs={r_{1,1}[2, 3], γ_{1,0}[1]}At gpair with Bs={r_{1,1}[1, 3], γ_{1,0}[2]}At gpair with Bs={r_{1,1}[1, 2], γ_{1,0}[3]}At gpair with Bs={r_{1,1}[1, 2], γ_{1,0}[2]}

(Alt) Out[]:=

$$\left\{ -\frac{1 - 3T + T^2}{T}, 0, \frac{(1 - 3T + T^2)(1 - T + T^2)}{T^2} \right\}$$

(Alt) In[]:=

Simplify[z2[[3]] - (z2[[3]] /. T → T⁻¹)]

(Alt) Out[]:=

0

(Alt) In[]:=

$$\mathbf{cb}_{2,10} = \frac{3 - 4 \mathbf{ca}_{2,1}}{2}$$

(Alt) Out[]:=

$$\frac{3}{2}$$

(Alt) In[]:=

```

z3 = Factor@ $\rho_2$ [Knot[8, 17]]
At gpair with Bs={r1,-1[1, 3], r1,-1[2, 4]}
At gpair with Bs={r1,-1[3, 2], r1,-1[4, 1]}
At gpair with Bs={r1,-1[1, 2], r1,1[3, 4]}
At gpair with Bs={r1,-1[2, 1], r1,1[3, 4]}
At gpair with Bs={r1,1[1, 3], r1,1[2, 4]}
At gpair with Bs={r1,-1[1, 2], r1,1[4, 3]}
At gpair with Bs={r1,-1[1, 3], r1,1[4, 2]}
At gpair with Bs={r1,-1[2, 1], r1,1[4, 3]}
At gpair with Bs={r1,-1[3, 1], r1,1[4, 2]}
At gpair with Bs={r1,1[1, 2], r1,1[4, 3]}
At gpair with Bs={r1,1[1, 3], r1,1[4, 2]}
At gpair with Bs={r1,1[2, 4], r1,1[3, 1]}
At gpair with Bs={r1,1[2, 1], r1,1[2, 1]}
At gpair with Bs={r2,1[2, 1]}
At gpair with Bs={r1,1[3, 2],  $\gamma_{1,-1}$ [1]}
At gpair with Bs={r1,-1[2, 1],  $\gamma_{1,-1}$ [3]}
At gpair with Bs={r1,1[3, 1],  $\gamma_{1,-1}$ [2]}
At gpair with Bs={r1,1[3, 2],  $\gamma_{1,0}$ [1]}
At gpair with Bs={r1,1[2, 1],  $\gamma_{1,0}$ [1]}
At gpair with Bs={r1,1[3, 1],  $\gamma_{1,0}$ [2]}
At gpair with Bs={r1,1[2, 1],  $\gamma_{1,0}$ [2]}
At gpair with Bs={r1,1[2, 1],  $\gamma_{1,0}$ [3]}

```

(Alt) Out[]:=

$$\left\{ -\frac{1 - 4T + 8T^2 - 11T^3 + 8T^4 - 4T^5 + T^6}{T^3}, 0, -\frac{1}{T^{10}} (1 - T + T^2) (1 - 4T + 8T^2 - 11T^3 + 8T^4 - 4T^5 + T^6) (1 - 7T + 8T^2 + 46T^3 - 197T^4 + 377T^5 - 457T^6 + 377T^7 - 197T^8 + 46T^9 + 8T^{10} - 7T^{11} + T^{12}) \right\}$$

(Alt) In[]:=

```

Simplify[z3[[3]] - (z3[[3]] /. T -> T-1)]

```

(Alt) Out[]:=

0

(Alt) In[]:=

```
BeginProfile[]
Timing[z1 = ρ₂[Knot[10, 106]]]
PrintProfile[]
```

(Alt) Out[]:=

ProfileRoot
 At gpair with Bs={r₁,₁[3, 1], r₁,₁[4, 2]}

(Alt) Out[]:=

$$\left\{ 8.04688, \right. \\ \left. -\frac{(1-T+T^2)(-1+T-2T^2+T^3)(-1+2T-T^2+T^3)}{T^4}, -\frac{1}{T^8}(-1+T)^2(1-6T+20T^2-48T^3+82T^4-114T^5+134T^6-140T^7+134T^8-114T^9+82T^{10}-48T^{11}+20T^{12}-6T^{13}+T^{14}), \right. \\ \left. \frac{1}{2T^{16}}(1-16T+127T^2-676T^3+2735T^4-8980T^5+24938T^6-60420T^7+131072T^8-259992T^9+477614T^{10}-814576T^{11}+1282448T^{12}-1846716T^{13}+2411126T^{14}-2836312T^{15}+2995252T^{16}-2836312T^{17}+2411126T^{18}-1846716T^{19}+1282448T^{20}-814576T^{21}+477614T^{22}-259992T^{23}+131072T^{24}-60420T^{25}+24938T^{26}-8980T^{27}+2735T^{28}-676T^{29}+127T^{30}-16T^{31}+T^{32}) \right\}$$

(Alt) Out[]:=

```
ProfileRoot is root. Profiled time: 8.047
( 1) 0.015/ 8.047 above ρd
Pairing: called 1 times, time in 2.719/2.719
( 1) 2.719/ 2.719 under ρd
ExpandedMold: called 1 times, time in 2.063/2.063
( 1) 2.063/ 2.063 under ρd
Green: called 1 times, time in 1.797/1.797
( 1) 1.797/ 1.797 under ρd
Substitution: called 1 times, time in 1.453/1.453
( 1) 1.453/ 1.453 under ρd
ρd: called 1 times, time in 0.015/8.047
( 1) 0.015/ 8.047 under ProfileRoot
( 1) 1.797/ 1.797 above Green
( 1) 2.063/ 2.063 above ExpandedMold
( 1) 0/ 0 above Mold
( 1) 2.719/ 2.719 above Pairing
( 1) 1.453/ 1.453 above Substitution
Mold: called 1 times, time in 0./0.
( 1) 0/ 0 under ρd
```

(Alt) In[]:=

```
BeginProfile[]
Timing[z2 = ρ₂[Knot[12, NonAlternating, 369]]]
PrintProfile[]
```

(Alt) Out[]:=

ProfileRoot
 KnotTheory: Loading precomputed data in KnotTheory/12N.dts.

KnotTheory: The GaussCode to PD conversion was written by Siddarth Sankaran at the University of Toronto in the summer of 2005.

At gpair with Bs={r_{1,-1}[1, 4], r_{1,-1}[3, 2]}

At gpair with Bs={r_{1,-1}[3, 4], r_{1,1}[1, 2]}

At gpair with Bs={r_{1,-1}[4, 3], r_{1,1}[1, 2]}

At gpair with Bs={r_{1,-1}[2, 4], r_{1,1}[3, 1]}

At gpair with Bs={r_{1,-1}[3, 4], r_{1,1}[2, 1]}

At gpair with Bs={r_{1,-1}[4, 3], r_{1,1}[2, 1]}

At gpair with Bs={r_{1,-1}[4, 2], r_{1,1}[3, 1]}

At gpair with Bs={r_{1,1}[1, 4], r_{1,1}[3, 2]}

At gpair with Bs={r_{1,1}[2, 1], r_{1,1}[3, 4]}

At gpair with Bs={r_{1,1}[2, 1], r_{1,1}[4, 3]}

At gpair with Bs={r_{1,-1}[2, 3], γ_{1,-1}[1]}

At gpair with Bs={r_{1,-1}[3, 2], γ_{1,-1}[1]}

At gpair with Bs={r_{1,1}[2, 1], γ_{1,-1}[3]}

At gpair with Bs={r_{1,-1}[1, 3], γ_{1,1}[2]}

At gpair with Bs={r_{1,-1}[3, 1], γ_{1,1}[2]}

At gpair with Bs={r_{1,1}[1, 2], γ_{1,1}[3]}

At gpair with Bs={r_{1,1}[1, 3], γ_{1,1}[2]}

At gpair with Bs={r_{1,1}[2, 1], γ_{1,1}[3]}

At gpair with Bs={r_{1,1}[2, 1], γ_{1,1}[2]}

At gpair with Bs={r_{1,1}[2, 3], γ_{1,1}[1]}

At gpair with Bs={r_{1,1}[3, 2], γ_{1,1}[1]}

At gpair with Bs={γ_{1,-1}[1], γ_{1,1}[2]}

At gpair with Bs={γ_{1,-1}[2], γ_{1,1}[1]}

At gpair with Bs={γ_{1,0}[1], γ_{1,1}[2]}

At gpair with Bs={γ_{1,0}[2], γ_{1,1}[1]}

At gpair with Bs={γ_{1,1}[1], γ_{1,1}[1]}

At gpair with Bs={r_{1,-1}[1, 2], γ_{1,1}[2]}

At gpair with Bs={r_{1,-1}[2, 1], γ_{1,1}[3]}

At gpair with Bs={r_{1,1}[3, 1], γ_{1,1}[2]}

At gpair with Bs={γ_{1,1}[1], γ_{1,1}[2]}

At gpair with Bs={r_{1,-1}[1, 2], γ_{1,1}[3]}

At gpair with Bs={γ_{2,1}[1]}

(Alt) Out[]:=

$$\left\{ 66.9844, \right. \\ \left. \left\{ -\frac{(1-T+T^2)(-1+T-2T^2+T^3)(-1+2T-T^2+T^3)}{T^4}, -\frac{1}{T^8}(-1+T)^2(1-6T+20T^2-48T^3+82T^4-114T^5+134T^6-140T^7+134T^8-114T^9+82T^{10}-48T^{11}+20T^{12}-6T^{13}+T^{14}), \right. \right. \\ \left. \left. \frac{1}{2T^{16}}(1-16T+127T^2-668T^3+2631T^4-8324T^5+22282T^6-52780T^7+114992T^8-236376T^9+460598T^{10}-839688T^{11}+1404696T^{12}-2121524T^{13}+2862782T^{14}-3432312T^{15}+3647156T^{16}-3432312T^{17}+2862782T^{18}-2121524T^{19}+1404696T^{20}-839688T^{21}+460598T^{22}-236376T^{23}+114992T^{24}-52780T^{25}+22282T^{26}-8324T^{27}+2631T^{28}-668T^{29}+127T^{30}-16T^{31}+T^{32}) \right\} \right\}$$

(Alt) Out[]:=

```
ProfileRoot is root. Profiled time: 66.984
( 1) 0.047/ 66.984 above ρd
Pairing: called 1 times, time in 55.407/55.407
( 1) 55.407/ 55.407 under ρd
Substitution: called 1 times, time in 5.046/5.046
( 1) 5.046/ 5.046 under ρd
ExpandedMold: called 1 times, time in 3.75/3.75
( 1) 3.750/ 3.750 under ρd
Green: called 1 times, time in 2.734/2.734
( 1) 2.734/ 2.734 under ρd
ρd: called 1 times, time in 0.047/66.984
( 1) 0.047/ 66.984 under ProfileRoot
( 1) 2.734/ 2.734 above Green
( 1) 3.750/ 3.750 above ExpandedMold
( 1) 0/ 0 above Mold
( 1) 55.407/ 55.407 above Pairing
( 1) 5.046/ 5.046 above Substitution
Mold: called 1 times, time in 0./0.
( 1) 0/ 0 under ρd
```

(Alt) In[]:=

Simplify[Thread[z1 == z2]]

(Alt) Out[]:=

$$\left\{ \text{True, True,} \right. \\ \left. \frac{1}{T}(-1+T)(1-T+T^2)(1-6T+16T^2-23T^3+9T^4+47T^5-141T^6+231T^7-272T^8+231T^9-141T^{10}+47T^{11}+9T^{12}-23T^{13}+16T^{14}-6T^{15}+T^{16}) == 0 \right\}$$

(Alt) In[]:=

```
TableForm[Table[Factor@Join[{K[[1]]K[[2]]}, z = ρ2[K],
{Coefficient[z[[3]], ca2,1], z[[1]], PolynomialRemainder[z[[3]], z[[1]], T}],
{K, AllKnots[{3, 7}]}], TableAlignments -> Center]
```

Coefficient: 0 is not a valid variable.

Coefficient: 0 is not a valid variable.

Power: Infinite expression $\frac{1}{0}$ encountered.

Coefficient: 0 is not a valid variable.

General: Further output of Coefficient::ivar will be suppressed during this calculation.

At gpair with Bs={r_{1,-1}[2, 1], r_{1,-1}[3, 4]}

At gpair with Bs={r_{1,-1}[2, 1], r_{1,-1}[4, 3]}

Power: Infinite expression $\frac{1}{0}$ encountered.

At gpair with Bs={r_{1,-1}[1, 4], r_{1,-1}[2, 3]}

At gpair with Bs={r_{1,1}[2, 3], r_{1,1}[4, 1]}

At gpair with Bs={r_{1,1}[3, 2], r_{1,1}[4, 1]}

(Alt) Out[]//TableForm=

3 ₁	$\frac{1-T+T^2}{T}$	$\frac{(-1+T)^2 (1+T^2)}{T^2}$	
4 ₁	$-\frac{1-3T+T^2}{T}$	0	
5 ₁	$\frac{1-T+T^2-T^3+T^4}{T^2}$	$\frac{(-1+T)^2 (1+T^2) (2+T^2+2T^4)}{T^4}$	$\frac{4-16T+35T^2-60T^3}{T^4}$
5 ₂	$\frac{2-3T+2T^2}{T}$	$\frac{(-1+T)^2 (5-4T+5T^2)}{T^2}$	
6 ₁	$-\frac{(-2+T)(-1+2T)}{T}$	$\frac{(-1+T)^2 (1-4T+T^2)}{T^2}$	
6 ₂	$-\frac{1-3T+3T^2-3T^3+T^4}{T^2}$	$\frac{(-1+T)^2 (1-4T+4T^2-4T^3+4T^4-4T^5+T^6)}{T^4}$	$\frac{1-12T+62T^2-180T^3+3}{T^4}$
6 ₃	$\frac{1-3T+5T^2-3T^3+T^4}{T^2}$	0	
7 ₁	$\frac{1-T+T^2-T^3+T^4-T^5+T^6}{T^3}$	$\frac{(-1+T)^2 (1+T^2) (3+2T^2+4T^4+2T^6+3T^8)}{T^6}$	$\frac{9-36T+83T^2-152T^3+238T^4-336T^5+434T^6-556T^7+719T^8}{T^6}$
7 ₂	$\frac{3-5T+3T^2}{T}$	$\frac{2(-1+T)^2 (7-8T+7T^2)}{T^2}$	
7 ₃	$\frac{2-3T+3T^2-3T^3+2T^4}{T^2}$	$-\frac{(-1+T)^2 (9-8T+16T^2-12T^3+16T^4-8T^5+9T^6)}{T^4}$	$\frac{82-472T+1409T^2-2996T^3+5190T^4}{T^4}$
7 ₄	$\frac{4-7T+4T^2}{T}$	$-\frac{8(-1+T)^2 (3-4T+3T^2)}{T^2}$	
7 ₅	$\frac{2-4T+5T^2-4T^3+2T^4}{T^2}$	$\frac{(-1+T)^2 (9-16T+29T^2-28T^3+29T^4-16T^5+9T^6)}{T^4}$	$\frac{82-616T+2412T^2-6560T^3+13875T^4-2}{T^4}$
7 ₆	$-\frac{1-5T+7T^2-5T^3+T^4}{T^2}$	$\frac{(-1+T)^2 (1-8T+19T^2-20T^3+19T^4-8T^5+T^6)}{T^4}$	$\frac{1-20T+175T^2-880T^3+2923T^4}{T^4}$
7 ₇	$\frac{1-5T+9T^2-5T^3+T^4}{T^2}$	$-\frac{(-1+T)^2 (3-8T+3T^2)}{T^2}$	$\frac{1-20T+4}{T^2}$

(Alt) In[]:=

```
OverbayP2Data = Get["C:\\drorbn\\AcademicPensieve\\People\\Overbay\\OverbayP2Data.m"];  
P2[K_] := K /. OverbayP2Data;
```

(Alt) In[]:=

```
TableForm[Table[Factor@Join[{K[[1]]K[[2]]}, z = rho2[K],
  {p2[K],  $\frac{\text{Coefficient}[z[[3]], ca_{2,1}]}{z[[2]]}$ , z[[1]], PolynomialRemainder[z[[3]], z[[1]], T]}],
  {K, AllKnots[{3, 7]}}, TableAlignments -> Center]
```

Coefficient: 0 is not a valid variable.

Coefficient: 0 is not a valid variable.

Power: Infinite expression $\frac{1}{0}$ encountered.

Coefficient: 0 is not a valid variable.

General: Further output of Coefficient::ivar will be suppressed during this calculation.

Power: Infinite expression $\frac{1}{0}$ encountered.

(Alt) Out[]//TableForm=

3 ₁	$\frac{1-T+T^2}{T}$	$\frac{(-1+T)^2(1+T^2)}{T^2}$	
4 ₁	$-\frac{1-3T+T^2}{T}$	0	
5 ₁	$\frac{1-T+T^2-T^3+T^4}{T^2}$	$\frac{(-1+T)^2(1+T^2)(2+T^2+2T^4)}{T^4}$	$\frac{4-16T+35T^2-60T}{T^4}$
5 ₂	$\frac{2-3T+2T^2}{T}$	$\frac{(-1+T)^2(5-4T+5T^2)}{T^2}$	
6 ₁	$-\frac{(-2+T)(-1+2T)}{T}$	$\frac{(-1+T)^2(1-4T+T^2)}{T^2}$	
6 ₂	$-\frac{1-3T+3T^2-3T^3+T^4}{T^2}$	$\frac{(-1+T)^2(1-4T+4T^2-4T^3+4T^4-4T^5+T^6)}{T^4}$	$\frac{1-12T+62T^2-180T^3+3}{T^4}$
6 ₃	$\frac{1-3T+5T^2-3T^3+T^4}{T^2}$	0	
7 ₁	$\frac{1-T+T^2-T^3+T^4-T^5+T^6}{T^3}$	$\frac{(-1+T)^2(1+T^2)(3+2T^2+4T^4+2T^6+3T^8)}{T^6}$	$\frac{9-36T+83T^2-152T^3+238T^4-336T^5+434T^6-556T^7+719T}{T^6}$
7 ₂	$\frac{3-5T+3T^2}{T}$	$\frac{2(-1+T)^2(7-8T+7T^2)}{T^2}$	
7 ₃	$\frac{2-3T+3T^2-3T^3+2T^4}{T^2}$	$-\frac{(-1+T)^2(9-8T+16T^2-12T^3+16T^4-8T^5+9T^6)}{T^4}$	$\frac{82-472T+1409T^2-2996T^3+5190T^4}{T^4}$
7 ₄	$\frac{4-7T+4T^2}{T}$	$-\frac{8(-1+T)^2(3-4T+3T^2)}{T^2}$	
7 ₅	$\frac{2-4T+5T^2-4T^3+2T^4}{T^2}$	$\frac{(-1+T)^2(9-16T+29T^2-28T^3+29T^4-16T^5+9T^6)}{T^4}$	$\frac{82-616T+2412T^2-6560T^3+13875T^4-2}{T^4}$
7 ₆	$-\frac{1-5T+7T^2-5T^3+T^4}{T^2}$	$\frac{(-1+T)^2(1-8T+19T^2-20T^3+19T^4-8T^5+T^6)}{T^4}$	$\frac{1-20T+175T^2-880T^3+2923T^4}{T^4}$
7 ₇	$\frac{1-5T+9T^2-5T^3+T^4}{T^2}$	$-\frac{(-1+T)^2(3-8T+3T^2)}{T^2}$	$\frac{1-20T+}{T^2}$

(Alt) In[]:=

```

E[L_, Q_, P_]$_k := E[L, Q, Series[Normal@P, {ϵ, 0, $k}]];
E_{d→r}[L_, Q_, P_]$_k := E_{d→r} @@ E[L, Q, P]$_k;
E3@E[ω_, L_, Q_, Ps_] := Factor /@ E[L, ω-1 Q, ω-1 (ω-4 ϵ)-1+Range@Length@Ps.Ps]$_k;
E4@E[L_, Q_, P_] := Module[
  {ω = Normal[P]-1 /. ϵ → 0, Ps = CoefficientList[P, ϵ]},
  Factor /@ E[ω, L, ω Q, ω-3+4 Range@Length@Ps Ps]];
E3@E_{sp___}[as___] := E3@E[as] /. E → E_{sp};
E4@E_{sp___}[as___] := E4@E[as] /. E → E_{sp};

```

(Alt) In[]:=

```

$k = 2;
Clear[QP, ω];
QP[Knot[n_, k_]] := QP[Knot[n, k]] = Collect[Module[{fname},
  fname = "../Projects/PG/Data/" <> ToString[n] <> "_" <> ToString[k] <> ".m";
  Collect[E3[Get[fname][[2, 2]][[3]] // Normal, ϵ, Simplify]
], ϵ, Factor];
ω[K_Knot] := ω[K] = Factor[(QP@K /. ϵ → 0)-1];
c_{r,d}[K_Knot] :=
  Factor[SeriesCoefficient[QP[K], {y, 0, 0}, {ϵ, 0, k}, {a, 0, d}] ω[K]1+2k-d]

```

(Alt) In[]:=

```
c_{1,1}[Knot[3, 1]]
```

(Alt) Out[]:=

$$\frac{2(-1+T)(1+T)}{T}$$

(Alt) In[]:=

```

Clear[ρ1, ρ2];
ρ1[K_Knot] := ρ1[K] = Factor[ $\frac{T(-c_{1,0}[K] + \omega[K] T \partial_T \omega[K])}{(T-1)^2}$ ];
ρ2[K_Knot] := ρ2[K] = Factor[-2 c_{2,0}[K] + ω[K] c_{2,1}[K]];
plus[ε_] := Expand[ε] /. Tn -> 0; n < 0 => 0;

```


(Alt) In[]:=

```
TableForm[Table[Factor@Join[{K[[1]]K[[2]], ρ1[K], ρ2[K]}, z = ρ2[K] /. ca2,1 → 0;
{

$$\frac{z[[2]]}{(T-1)^2/T}, z[[3]]}], \{K, \text{AllKnots}[\{3, 7\}]\}, \text{TableAlignments} \rightarrow \text{Center}]$$

```

(Alt) Out[]//TableForm=

3 ₁	$\frac{1+T^2}{T}$	$\frac{3-12T+26T^2-38T^3+26T^4}{T^3}$
4 ₁	0	$\frac{(1-3T+T^2)(1-16T^2+26T^3-10T^4)}{T^4}$
5 ₁	$\frac{(1+T^2)(2+T^2+2T^4)}{T^3}$	$\frac{5-20T+55T^2-120T^3+217T^4-338T^5+450T^6-510T^7+450T^8}{T^7}$
5 ₂	$\frac{5-4T+5T^2}{T}$	$-\frac{10-120T+487T^2-1054T^3+1362T^4-1095T^5}{T^4}$
6 ₁	$\frac{1-4T+T^2}{T}$	$\frac{14-16T-293T^2+1098T^3-1598T^4+1095T^5}{T^4}$
6 ₂	$\frac{1-4T+4T^2-4T^3+4T^4-4T^5+T^6}{T^3}$	$\frac{3-21T+49T^2+15T^3-433T^4+1543T^5-3431T^6+5482T^7-6410T^8+5482T^9}{T^8}$
6 ₃	0	$\frac{(1-3T+5T^2-3T^3+T^4)(4-21T+38T^2+28T^3-284T^4+642T^5-81T^6)}{T^8}$
7 ₁	$\frac{(1+T^2)(3+2T^2+4T^4+2T^6+3T^8)}{T^5}$	$\frac{7-28T+77T^2-168T^3+322T^4-560T^5+891T^6-1310T^7+1777T^8-2238T^9+2604T^{10}-2772T^{11}+2604T^{12}}{T^{11}}$
7 ₂	$\frac{2(7-8T+7T^2)}{T}$	$-\frac{129-1177T+4421T^2-9226T^3+11718T^4-92T^5}{T^4}$
7 ₃	$-\frac{9-8T+16T^2-12T^3+16T^4-8T^5+9T^6}{T^3}$	$-\frac{18-208T+917T^2-2666T^3+6049T^4-11283T^5+17671T^6-23356T^7+25736T^8-2335T^9}{T^8}$
7 ₄	$-\frac{8(3-4T+3T^2)}{T}$	$-\frac{2(176-1808T+7189T^2-15350T^3+19594T^4-1T^5)}{T^4}$
7 ₅	$\frac{9-16T+29T^2-28T^3+29T^4-16T^5+9T^6}{T^3}$	$-\frac{18-264T+1548T^2-5680T^3+15107T^4-31152T^5+51476T^6-69252T^7+76414T^8-6925T^9}{T^8}$
7 ₆	$\frac{1-8T+19T^2-20T^3+19T^4-8T^5+T^6}{T^3}$	$\frac{3-35T+128T^2+105T^3-2610T^4+11225T^5-28031T^6+47186T^7-55946T^8+47186T^9}{T^8}$
7 ₇	$-\frac{3-8T+3T^2}{T}$	$\frac{4-55T+310T^2-805T^3+86T^4+6349T^5-22686T^6+43610T^7-53622T^8+43610T^9}{T^8}$

(Alt) In[]:=

```
MatrixForm[tab = Table[z = ρ2[K]; Ω = z[[1]];
Factor@Join[{ρ1[K], ρ2[K]}, {
P2[K], Ω  $\frac{T^2}{T^2-1}$  ∂Tρ1[K],  $\frac{T^2}{T^2-1}$  ∂TΩ, Ω  $\frac{T^2}{T^2-1}$  ∂TΩ,
Ω2  $\frac{T^2}{T^2-1}$  ∂TΩ, Ω3 Factor[ $\frac{T^2}{T^2-1}$  ∂TΩ], Ω2 ( $\frac{T^2}{T^2-1}$  ∂TΩ)2, Ω Factor[ $\frac{T^2}{T^2-1}$  ∂TΩ] × ρ1[K],
Ω3 Factor[ $\frac{(T ∂_T (T ∂_T Ω)) - T \frac{T^2+1}{T^2-1} ∂_T Ω}{2 \frac{(-1+T)^2 (1+T)^2}{T^2}}$ ], ρ1[K] Ω, Ω3,  $\frac{z[[2]]}{(T-1)^2/T}$ ,
z[[3]] /. ca2,1 → 0, Coefficient[z[[3]], ca2,1]}], {K, AllKnots[{3, 7}]}]]
```

NullSpace[tab /. T → 100]

Coefficient: 0 is not a valid variable.

Coefficient: 0 is not a valid variable.

Coefficient: 0 is not a valid variable.

General: Further output of Coefficient::ivar will be suppressed during this calculation.

(Alt) Out[]//MatrixForm=

$$\begin{matrix}
 \frac{1+T^2}{T} & \frac{3-12 T+26 T^2-38 T^3+26 T^4-12 T^5+3 T^6}{T^3} \\
 0 & \frac{(1-3 T+T^2)(1-16 T^2+26 T^3-16 T^4+T^6)}{T^4} \\
 \frac{(1+T^2)(2+T^2+2 T^4)}{T^3} & \frac{5-20 T+55 T^2-120 T^3+217 T^4-338 T^5+450 T^6-510 T^7+450 T^8-338 T^9+21}{T^7} \\
 \frac{5-4 T+5 T^2}{T} & - \frac{10-120 T+487 T^2-1054 T^3+1362 T^4-1054 T^5+487 T}{T^4} \\
 \frac{1-4 T+T^2}{T} & \frac{14-16 T-293 T^2+1098 T^3-1598 T^4+1098 T^5-293 T^6}{T^4} \\
 \frac{1-4 T+4 T^2-4 T^3+4 T^4-4 T^5+T^6}{T^3} & \frac{3-21 T+49 T^2+15 T^3-433 T^4+1543 T^5-3431 T^6+5482 T^7-6410 T^8+5482 T^9-3431 T^{10}+}{T^8} \\
 0 & \frac{(1-3 T+5 T^2-3 T^3+T^4)(4-21 T+38 T^2+28 T^3-284 T^4+642 T^5-818 T^6+642 T^7)}{T^8} \\
 \frac{(1+T^2)(3+2 T^2+4 T^4+2 T^6+3 T^8)}{T^5} & \frac{7-28 T+77 T^2-168 T^3+322 T^4-560 T^5+891 T^6-1310 T^7+1777 T^8-2238 T^9+2604 T^{10}-2772 T^{11}+2604 T^{12}-2238 T^{13}+}{T^{11}} \\
 \frac{2(7-8 T+7 T^2)}{T} & - \frac{129-1177 T+4421 T^2-9226 T^3+11718 T^4-9226 T^5+4421}{T^4} \\
 - \frac{9-8 T+16 T^2-12 T^3+16 T^4-8 T^5+9 T^6}{T^3} & - \frac{18-208 T+917 T^2-2666 T^3+6049 T^4-11283 T^5+17671 T^6-23356 T^7+25736 T^8-23356 T^9+17671}{T^8} \\
 - \frac{8(3-4 T+3 T^2)}{T} & - \frac{2(176-1808 T+7189 T^2-15350 T^3+19594 T^4-15350 T^5+71}{T^4} \\
 \frac{9-16 T+29 T^2-28 T^3+29 T^4-16 T^5+9 T^6}{T^3} & - \frac{18-264 T+1548 T^2-5680 T^3+15107 T^4-31152 T^5+51476 T^6-69252 T^7+76414 T^8-69252 T^9+51476}{T^8} \\
 \frac{1-8 T+19 T^2-20 T^3+19 T^4-8 T^5+T^6}{T^3} & \frac{3-35 T+128 T^2+105 T^3-2610 T^4+11225 T^5-28031 T^6+47186 T^7-55946 T^8+47186 T^9-28031}{T^8} \\
 - \frac{3-8 T+3 T^2}{T} & \frac{4-55 T+310 T^2-805 T^3+86 T^4+6349 T^5-22686 T^6+43610 T^7-53622 T^8+43610 T^9-22686}{T^8}
 \end{matrix}$$

Coefficient: 0 is not a valid variable.

Coefficient: 0 is not a valid variable.

Coefficient: 0 is not a valid variable.

General: Further output of Coefficient::ivar will be suppressed during this calculation.

(Alt) Out[]:=

$$\left\{ \left\{ 0, \frac{1}{2}, 0, 0, 0, 0, 0, \frac{10001}{200}, -\frac{99980001}{10000}, 0, \frac{99980001}{10000}, 0, 0, 0, 1, 0 \right\}, \right. \\
 \left. \left\{ -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0 \right\} \right\}$$

(Alt) In[]:=

Factor[tab - (tab /. T -> T^-1)]

Coefficient: 0 is not a valid variable.

Coefficient: 0 is not a valid variable.

Coefficient: 0 is not a valid variable.

General: Further output of Coefficient::ivar will be suppressed during this calculation.

(Alt) Out[]:=

$$\left\{ \left\{ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \right. \right. \\
 \left. \left. -\text{Coefficient} \left[\frac{1}{2} \left(1 + \frac{1}{T^8} - \frac{4}{T^7} + \frac{7}{T^6} - \frac{12}{T^5} + \frac{18}{T^4} - \frac{12}{T^3} + \frac{7}{T^2} - \frac{4}{T} \right) T^4, 0 \right] + \right. \right.$$

$$\begin{aligned}
 & \text{Coefficient} \left[\frac{1 - 4 T + 7 T^2 - 12 T^3 + 18 T^4 - 12 T^5 + 7 T^6 - 4 T^7 + T^8}{2 T^4}, \theta \right], \\
 & \left\{ \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \right. \\
 & \left. - \text{Coefficient} \left[\left(1 + \frac{1}{T^2} - \frac{3}{T} \right) \left(1 + \frac{1}{T^2} - \frac{1}{T} \right) T^2, \theta \right] + \text{Coefficient} \left[\frac{(1 - 3 T + T^2)(1 - T + T^2)}{T^2}, \theta \right], \right. \\
 & \left. \left\{ \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, - \text{Coefficient} \left[\frac{1}{2} \left(4 + \frac{4}{T^{16}} - \frac{16}{T^{15}} + \frac{35}{T^{14}} - \frac{60}{T^{13}} + \right. \right. \right. \right. \\
 & \left. \left. \left. \frac{85}{T^{12}} - \frac{120}{T^{11}} + \frac{170}{T^{10}} - \frac{220}{T^9} + \frac{250}{T^8} - \frac{220}{T^7} + \frac{170}{T^6} - \frac{120}{T^5} + \frac{85}{T^4} - \frac{60}{T^3} + \frac{35}{T^2} - \frac{16}{T} \right) T^8, \theta \right] + \right. \\
 & \left. \text{Coefficient} \left[\frac{1}{2 T^8} (4 - 16 T + 35 T^2 - 60 T^3 + 85 T^4 - 120 T^5 + 170 T^6 - 220 T^7 + 250 T^8 - 220 T^9 + \right. \right. \\
 & \left. \left. 170 T^{10} - 120 T^{11} + 85 T^{12} - 60 T^{13} + 35 T^{14} - 16 T^{15} + 4 T^{16}), \theta \right], \left\{ \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \right. \right. \\
 & \left. \left. \theta, \theta, \theta, \theta, - \text{Coefficient} \left[\frac{1}{2} \left(26 + \frac{26}{T^8} - \frac{144}{T^7} + \frac{387}{T^6} - \frac{688}{T^5} + \frac{842}{T^4} - \frac{688}{T^3} + \frac{387}{T^2} - \frac{144}{T} \right) T^4, \theta \right] + \right. \\
 & \left. \text{Coefficient} \left[\frac{26 - 144 T + 387 T^2 - 688 T^3 + 842 T^4 - 688 T^5 + 387 T^6 - 144 T^7 + 26 T^8}{2 T^4}, \theta \right], \right. \\
 & \left. \left\{ \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \right. \right. \\
 & \left. \left. - \text{Coefficient} \left[\frac{1}{2} \left(2 + \frac{2}{T^8} - \frac{24}{T^7} + \frac{129}{T^6} - \frac{328}{T^5} + \frac{438}{T^4} - \frac{328}{T^3} + \frac{129}{T^2} - \frac{24}{T} \right) T^4, \theta \right] + \right. \\
 & \left. \text{Coefficient} \left[\frac{2 - 24 T + 129 T^2 - 328 T^3 + 438 T^4 - 328 T^5 + 129 T^6 - 24 T^7 + 2 T^8}{2 T^4}, \theta \right], \right. \\
 & \left. \left\{ \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, - \text{Coefficient} \left[\frac{1}{2} \left(1 + \frac{1}{T^{16}} - \frac{12}{T^{15}} + \frac{62}{T^{14}} - \frac{180}{T^{13}} + \right. \right. \right. \right. \\
 & \left. \left. \left. \frac{354}{T^{12}} - \frac{592}{T^{11}} + \frac{1007}{T^{10}} - \frac{1576}{T^9} + \frac{1870}{T^8} - \frac{1576}{T^7} + \frac{1007}{T^6} - \frac{592}{T^5} + \frac{354}{T^4} - \frac{180}{T^3} + \frac{62}{T^2} - \frac{12}{T} \right) T^8, \theta \right] + \right. \\
 & \left. \text{Coefficient} \left[\frac{1}{2 T^8} (1 - 12 T + 62 T^2 - 180 T^3 + 354 T^4 - 592 T^5 + 1007 T^6 - 1576 T^7 + 1870 T^8 - \right. \right. \\
 & \left. \left. 1576 T^9 + 1007 T^{10} - 592 T^{11} + 354 T^{12} - 180 T^{13} + 62 T^{14} - 12 T^{15} + T^{16}), \theta \right], \right. \\
 & \left. \left\{ \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \right. \right. \\
 & \left. \left. - \text{Coefficient} \left[- \left(\left(1 + \frac{1}{T^6} - \frac{11}{T^4} + \frac{19}{T^3} - \frac{11}{T^2} \right) \left(1 + \frac{1}{T^4} - \frac{3}{T^3} + \frac{5}{T^2} - \frac{3}{T} \right) \left(1 + \frac{1}{T^2} - \frac{1}{T} \right) T^6, \theta \right] + \right. \right. \\
 & \left. \left. \text{Coefficient} \left[- \frac{(1 - T + T^2)(1 - 3 T + 5 T^2 - 3 T^3 + T^4)(1 - 11 T^2 + 19 T^3 - 11 T^4 + T^6)}{T^6}, \theta \right], \right. \right. \\
 & \left. \left. \left\{ \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \theta, \right. \right. \right. \\
 & \left. \left. - \text{Coefficient} \left[\frac{1}{2} \left(9 + \frac{9}{T^{24}} - \frac{36}{T^{23}} + \frac{83}{T^{22}} - \frac{152}{T^{21}} + \frac{238}{T^{20}} - \frac{336}{T^{19}} + \frac{434}{T^{18}} - \frac{556}{T^{17}} + \frac{719}{T^{16}} - \frac{912}{T^{15}} + \frac{1117}{T^{14}} - \frac{1288}{T^{13}} + \right. \right. \right. \\
 & \left. \left. \left. \frac{1372}{T^{12}} - \frac{1288}{T^{11}} + \frac{1117}{T^{10}} - \frac{912}{T^9} + \frac{719}{T^8} - \frac{556}{T^7} + \frac{434}{T^6} - \frac{336}{T^5} + \frac{238}{T^4} - \frac{152}{T^3} + \frac{83}{T^2} - \frac{36}{T} \right) T^{12}, \theta \right] + \right. \\
 & \left. \left. \left. \right. \right. \right.
 \end{aligned}$$

$$6340 T^7 + 3302 T^8 - 1064 T^9 + 199 T^{10} - 20 T^{11} + T^{12}, \theta \}}\}}\}$$

(Alt) In[]:=

```
MatrixForm[tab = Table[z = ρ2[K]; Ω = z[[1]];
```

```
Factor@Join[{ρ2[K]}, {Ω^3 Factor[T^2 / (T^2 - 1) ∂T Ω], Ω^2 (T^2 / (T^2 - 1) ∂T Ω)^2,
```

```
Ω^3 Factor[(T ∂T (T ∂T Ω)) - T T^{2+1} / (T^2 - 1) ∂T Ω] / (2 (-1+T)^2 (1+T)^2 / T^2), z[[3]] /. ca_{2,1} -> θ}], {K, AllKnots[{3, 7}]}]]
```

```
NullSpace[tab]
```

(Alt) Out[]:=

$$\left\{ \begin{aligned} & \frac{3 - 12 T + 26 T^2 - 38 T^3 + 26 T^4 - 12 T^5 + 3 T^6}{T^3} \\ & \frac{(1 - 3 T + T^2) (1 - 16 T^2 + 26 T^3 - 16 T^4 + T^6)}{T^4} \\ & \frac{5 - 20 T + 55 T^2 - 120 T^3 + 217 T^4 - 338 T^5 + 450 T^6 - 510 T^7 + 450 T^8 - 338 T^9 + 217 T^{10} - 120 T^{11} + 55 T^{12} - 20 T^{13} + 5 T^{14}}{T^7} \\ & - \frac{10 - 120 T + 487 T^2 - 1054 T^3 + 1362 T^4 - 1054 T^5 + 487 T^6 - 120 T^7 + 10 T^8}{T^4} \\ & \frac{14 - 16 T - 293 T^2 + 1098 T^3 - 1598 T^4 + 1098 T^5 - 293 T^6 - 16 T^7 + 14 T^8}{T^4} \\ & \frac{3 - 21 T + 49 T^2 + 15 T^3 - 433 T^4 + 1543 T^5 - 3431 T^6 + 5482 T^7 - 6410 T^8 + 5482 T^9 - 3431 T^{10} + 1543 T^{11} - 433 T^{12} + 15 T^{13} + 49 T^{14} - 21 T^{15} + 3}{T^8} \\ & \frac{(1 - 3 T + 5 T^2 - 3 T^3 + T^4) (4 - 21 T + 38 T^2 + 28 T^3 - 284 T^4 + 642 T^5 - 818 T^6 + 642 T^7 - 284 T^8 + 28 T^9 + 38 T^{10} - 21 T^{11} + 4 T^{12})}{T^8} \\ & \frac{7 - 28 T + 77 T^2 - 168 T^3 + 322 T^4 - 560 T^5 + 891 T^6 - 1310 T^7 + 1777 T^8 - 2238 T^9 + 2604 T^{10} - 2772 T^{11} + 2604 T^{12} - 2238 T^{13} + 1777 T^{14} - 1310 T^{15} + 891 T^{16} - 560 T^{17} + 322}{T^{11}} \\ & - \frac{129 - 1177 T + 4421 T^2 - 9226 T^3 + 11718 T^4 - 9226 T^5 + 4421 T^6 - 1177 T^7 + 129 T^8}{T^4} \\ & - \frac{18 - 208 T + 917 T^2 - 2666 T^3 + 6049 T^4 - 11283 T^5 + 17671 T^6 - 23356 T^7 + 25736 T^8 - 23356 T^9 + 17671 T^{10} - 11283 T^{11} + 6049 T^{12} - 2666 T^{13} + 917 T^{14}}{T^8} \\ & - \frac{2 (176 - 1808 T + 7189 T^2 - 15350 T^3 + 19594 T^4 - 15350 T^5 + 7189 T^6 - 1808 T^7 + 176 T^8)}{T^4} \\ & - \frac{18 - 264 T + 1548 T^2 - 5680 T^3 + 15107 T^4 - 31152 T^5 + 51476 T^6 - 69252 T^7 + 76414 T^8 - 69252 T^9 + 51476 T^{10} - 31152 T^{11} + 15107 T^{12} - 5680 T^{13} + 1548}{T^8} \\ & \frac{3 - 35 T + 128 T^2 + 105 T^3 - 2610 T^4 + 11225 T^5 - 28031 T^6 + 47186 T^7 - 55946 T^8 + 47186 T^9 - 28031 T^{10} + 11225 T^{11} - 2610 T^{12} + 105 T^{13} + 128 T^{14}}{T^8} \\ & \frac{4 - 55 T + 310 T^2 - 805 T^3 + 86 T^4 + 6349 T^5 - 22686 T^6 + 43610 T^7 - 53622 T^8 + 43610 T^9 - 22686 T^{10} + 6349 T^{11} + 86 T^{12} - 805 T^{13} + 310 T^{14} - 55}{T^8} \end{aligned} \right\}$$

(Alt) Out[]:=

$$\left\{ \left\{ \frac{1}{2}, -\frac{1 - T^2}{2 T}, -\frac{1 - 2 T^2 + T^4}{T^2}, -\frac{-1 + 2 T^2 - T^4}{T^2}, 1 \right\} \right\}$$

(Alt) In[]:=

```
MatrixForm[tab = Table[z = ρ2[K]; Ω = z[[1]];
Factor@Join[{ρ2[K]}, {Ω^3 Factor[

$$\frac{T^2}{T^2-1} \partial_T \Omega$$
], Ω^2 
$$\left(\frac{T^2}{T^2-1} \partial_T \Omega\right)^2$$
,
Ω^3 Factor[

$$\frac{(T \partial_T (T \partial_T \Omega)) - T \frac{T^2+1}{T^2-1} \partial_T \Omega}{\frac{2(-1+T)^2(1+T)^2}{T^2}}$$
], z[[3]] /. ca_{2,1} → 0}], {K, AllKnots[{3, 8]}]]]
```

NullSpace[tab]

(Alt) Out[]:=MatrixForm=

$$\begin{aligned} & \frac{3-12T+26T^2-38T^3+26T^4-12T^5+3T^6}{T^3} \\ & \frac{(1-3T+T^2)(1-16T^2+26T^3-16T^4+T^6)}{T^4} \\ & \frac{5-20T+55T^2-120T^3+217T^4-338T^5+450T^6-510T^7+450T^8-338T^9+217T^{10}-120T^{11}}{T^7} \\ & - \frac{10-120T+487T^2-1054T^3+1362T^4-1054T^5+487T^6-120T^7+10T^8}{T^4} \\ & \frac{14-16T-293T^2+1098T^3-1598T^4+1098T^5-293T^6-16T^7+14T^8}{T^4} \\ & \frac{3-21T+49T^2+15T^3-433T^4+1543T^5-3431T^6+5482T^7-6410T^8+5482T^9-3431T^{10}+1543T^{11}-3T^{12}}{T^8} \\ & \frac{(1-3T+5T^2-3T^3+T^4)(4-21T+38T^2+28T^3-284T^4+642T^5-818T^6+642T^7-284T^8+12T^9-12T^{10}+T^{11})}{T^8} \\ & \frac{7-28T+77T^2-168T^3+322T^4-560T^5+891T^6-1310T^7+1777T^8-2238T^9+2604T^{10}-2772T^{11}+2604T^{12}-2238T^{13}+1777T^{14}-7T^{15}}{T^{11}} \\ & - \frac{129-1177T+4421T^2-9226T^3+11718T^4-9226T^5+4421T^6-1177T^7}{T^4} \\ & - \frac{18-208T+917T^2-2666T^3+6049T^4-11283T^5+17671T^6-23356T^7+25736T^8-23356T^9+17671T^{10}-11283T^{11}+18T^{12}}{T^8} \\ & - \frac{2(176-1808T+7189T^2-15350T^3+19594T^4-15350T^5+7189T^6-180T^7)}{T^4} \\ & - \frac{18-264T+1548T^2-5680T^3+15107T^4-31152T^5+51476T^6-69252T^7+76414T^8-69252T^9+51476T^{10}-31152T^{11}+18T^{12}}{T^8} \\ & \frac{3-35T+128T^2+105T^3-2610T^4+11225T^5-28031T^6+47186T^7-55946T^8+47186T^9-28031T^{10}+11225T^{11}-3T^{12}}{T^8} \\ & \frac{4-55T+310T^2-805T^3+86T^4+6349T^5-22686T^6+43610T^7-53622T^8+43610T^9-22686T^{10}+6349T^{11}-4T^{12}}{T^8} \\ & \frac{42+215T-2542T^2+7562T^3-10542T^4+7562T^5-2542T^6+215T^7}{T^4} \\ & \frac{(-1+T)^2(5-29T+56T^2+2T^3-301T^4+1056T^5-2546T^6+4983T^7-8301T^8+12010T^9-15200T^{10}+16578T^{11}-15200T^{12}+12010T^{13}-8T^{14})}{T^{12}} \\ & \frac{2(4-9T+4T^2)(28+35T-438T^2+742T^3-438T^4+35T^5+28T^6)}{T^4} \\ & \frac{54-344T+865T^2-650T^3-2723T^4+12243T^5-28461T^6+45792T^7-53540T^8+45792T^9-28461T^{10}+12243T^{11}-54T^{12}}{T^8} \\ & \frac{5-39T+128T^2-182T^3-274T^4+2476T^5-8642T^6+21517T^7-42924T^8+71719T^9-102448T^{10}+126480T^{11}-135628T^{12}+126480T^{13}-102448T^{14}+71T^{15}}{T^{12}} \\ & \frac{38-216T+112T^2+2880T^3-14787T^4+42444T^5-85415T^6+128406T^7-146916T^8+128406T^9-85415T^{10}+424T^{11}}{T^8} \\ & \frac{8-75T+343T^2-979T^3+1821T^4-1782T^5-1623T^6+12083T^7-33001T^8+64599T^9-101194T^{10}+131404T^{11}-143216T^{12}+131404T^{13}-101194T^{14}+64T^{15}}{T^{12}} \\ & \frac{62-504T+1736T^2-2408T^3-3717T^4+26492T^5-68493T^6+113418T^7-133180T^8+113418T^9-68493T^{10}+264T^{11}}{T^8} \\ & \frac{(-1+T-2T^2+T^3)(-1+2T-T^2+T^3)(9-60T+192T^2-366T^3+335T^4+372T^5-2156T^6+4768T^7-7202T^8+8208T^9-7202T^{10}+47T^{11})}{T^{12}} \\ & \frac{(1-T+T^2)^4(9-12T+110T^2-174T^3+120T^4+895T^5+2200T^6+2665T^7+1254T^8+2665T^9+2200T^{10}+895T^{11})}{T^{12}} \end{aligned}$$

$$\frac{(1-T)^2 (38-236 T-66 T^2+5200 T^3-24311 T^4+62546 T^5-106347 T^6+126392 T^7-106347 T^8+62546 T^9-123-2128 T+15241 T^2-66120 T^3+199999 T^4-451912 T^5+792414 T^6-1101720 T^7+1228222 T^8-1101720 T^9+792414 T^{10}-8-100 T+598 T^2-2205 T^3+5292 T^4-7164 T^5-2380 T^6+43100 T^7-137314 T^8+291750 T^9-478742 T^{10}+636488 T^{11}-698666 T^{12}+636488 T^{13}-478742 T^{14}+291750 T^{15}-137314 T^{16}+43100 T^{17}-2380 T^{18}+478742 T^{19}-636488 T^{20}+478742 T^{21}-291750 T^{22}+636488 T^{23}-698666 T^{24}+636488 T^{25}-478742 T^{26}+291750 T^{27}-137314 T^{28}+43100 T^{29}-2380 T^{30}+5292 T^{31}-7164 T^{32}+5292 T^{33}-2205 T^{34}+598 T^{35}-100 T^{36}+8 T^{37}-4 T^{38}+T^{39})}{T^{12}}$$

(Alt) Out[]:=

$$\left\{ \left\{ \frac{1}{2}, -\frac{1-T^2}{2T}, -\frac{1-2T^2+T^4}{T^2}, -\frac{1+2T^2-T^4}{T^2}, 1 \right\} \right\}$$

(Alt) In[]:=

Factor [tab - (tab /. T -> T^-1)]

(Alt) Out[]:=

- { {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},
- {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},
- {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},
- {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},
- {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},
- {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0},
- {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}