

Pensieve header: Developing ρ_d .

Program

```
In[*]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\Oaxaca-2210"];
```

```
In[*]:= Once[<< KnotTheory` ; << Rot.m];
```

Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.

Read more at <http://katlas.org/wiki/KnotTheory>.

Loading Rot.m from <http://drorbn.net/la22/ap> to compute rotation numbers.

```
In[*]:= << "../..//Projects/Profile/Profile.m"
```

This is Profile.m of <http://www.drorbn.net/AcademicPensieve/Projects/Profile/>.

This version: April 2020. Original version: July 1994.

```
In[*]:= {ca1,2 = 1, ca1,10 = -1, ca2,1 = 0, cb2,10 = 3 / 2, cb3,10 = (7 - 12 ca3,1) / 6, ca3,1 = 0};
```

```
In[*]:= V@r1,φ[k_] := φ ( 1/2 - p̄_k x̄_k ); V@r2,φ[k_] := -φ²/2 p̄_k x̄_k; V@r3,φ[k_] := -1/6 φ³ p̄_k x̄_k;
```

```
In[*]:= V@r1,s[i_, j_] := 1/2 s (-1 + (p_i - p_j) x_i (2 + p_j ((-1 + T^s) x_i - 2 x_j)))
```

```
In[*]:= V@r2,1[i_, j_] := 1/12 (p_i - p_j) x_i (-6 +
p_j (x_i (3 - 9 T - 2 (-1 + T) (-2 p_i + (3 + T) p_j) x_i) + 6 (3 + (-p_i + (1 + T) p_j) x_i) x_j - 6 p_j x_j²))
```

```
In[*]:= V@r2,-1[i_, j_] :=
-1/(12 T²) (p_i - p_j) x_i (6 T² + p_j (2 (-1 + T) (2 T p_i - (1 + 3 T) p_j) x_i² + 6 T² x_j (-3 + p_j x_j) -
3 T x_i (-3 + T + 2 (-T p_i + (1 + T) p_j) x_j)))
```

```
In[*]:= V@r3,1[i_, j_] := 1/24 x_i (p_i³ p_j x_i² (3 (-1 + T) x_i - 4 x_j) +
p_i² p_j x_i (-3 (-4 + T + 3 T²) p_j x_i² + 12 x_j (3 - 2 p_j x_j) + 4 x_i (5 - 6 T + 7 T p_j x_j)) +
p_i (4 + p_j (x_i (2 (5 + 7 T) + p_j x_i (-64 + 68 T + 8 T² + (-1 + T) (13 + T (22 + T)) p_j x_i)) - 4 (7 +
p_j x_i (27 + 6 T + (-6 + T (17 + T)) p_j x_i)) x_j + 6 p_j (4 + (10 + T) p_j x_i) x_j² - 4 p_j² x_j³)) +
p_j (-4 + p_j (x_i (-2 (5 + 7 T) + p_j x_i (44 - 4 T (11 + 2 T) - (-1 + T) (4 + T (13 + T)) p_j x_i)) +
4 (7 + p_j x_i (6 (3 + T) + (-5 + T (10 + T)) p_j x_i)) x_j -
6 p_j (4 + (6 + T) p_j x_i) x_j² + 4 p_j² x_j³)))
```

```
In[ ]:=
V@r_{3,-1}[i_, j_] := - $\frac{1}{24 T^3} x_i (T^2 p_i^3 p_j x_i^2 (-3 (-1 + T) x_i - 4 T x_j) +$ 
  T p_i^2 p_j x_i (3 (-1 + T) (3 + 4 T) p_j x_i^2 + 12 T^2 x_j (3 - 2 p_j x_j) + 4 T x_i (-6 + 5 T + 7 p_j x_j)) +
  p_j (-4 T^3 + p_j ((-1 + T) (1 + T (13 + 4 T))) p_j^2 x_i^3 +
    4 T p_j x_i^2 (-2 + 11 (-1 + T) T + (1 - 5 (-2 + T) T) p_j x_j) + 4 T^3 x_j (7 + p_j x_j (-6 + p_j x_j)) +
    2 T^2 x_i (-7 - 5 T - 3 p_j x_j (-4 (1 + 3 T) + (1 + 6 T) p_j x_j))) +
  p_i (4 T^3 + p_j (-((-1 + T) (1 + T (22 + 13 T))) p_j^2 x_i^3) + 4 T p_j x_i^2
    (2 + (17 - 16 T) T + (-1 + T (-17 + 6 T)) p_j x_j) - 4 T^3 x_j (7 + p_j x_j (-6 + p_j x_j)) +
    2 T^2 x_i (7 + 5 T + 3 p_j x_j (-2 (2 + 9 T) + (1 + 10 T) p_j x_j)))));
```

```
In[ ]:=
{p*, x*, p-bar*, x-bar*} = {pi, xi, pi-bar, xi-bar}; (z_{i_})^* := (z^*)_i;
```

```
In[ ]:=
Zip_{ }[E_] := E;
Zip_{z_, zs_...}[E_] := (Collect[E // Zip_{zs}, z] /. f_ . z^{d_} -> (D[f, {z^*, d}])) /. z^* -> 0
```

```
In[ ]:=
gPair[fs_, w_] := gPair[fs, w] = PP_gPair [
  Print["Running gPair[" , fs, ", ", w, "] ..."];
  Collect[ZipJoin@@Table[{p_alpha, p_alpha-bar, x_alpha, x_alpha-bar}, {alpha, w}] [(Times @@ (V /@ fs))
    Exp[Sum[g_alpha, beta (pi_alpha + pi_alpha-bar) (xi_beta + xi_beta-bar), {alpha, w}, {beta, w}] - Sum[xi_alpha-bar pi_alpha, {alpha, w}]]],
  g_., Factor]
]
```

```

ρd_ [K_] := PP_ρd@Module [ {Cs, φ, n, A, s, i, j, k, Δ, G, d1, ρd1, ρd2, ρd3},
  PP"Green" [
    {Cs, φ} = Rot [K]; n = Length [Cs];
    A = IdentityMatrix [2 n + 1];
    Cases [Cs, {s_, i_, j_} => (A[[{i, j}, {i + 1, j + 1}]] += (
      -T^s T^s - 1
    ))];
    Δ = Factor [T^(-Total[φ] - Total[Cs[[All, 1]])/2] Det [A]];
    G = Factor@Inverse [A];
  ];
  ρd1 = PP_Mold@Exp [Total [Cases [Cs, {s_, i_, j_} => Sum [e^d1 r_{d1,s} [i, j], {d1, d}]]] +
    Sum [e^d1 γ_{d1,φ[[k]]} [k], {k, 2 n}, {d1, d}]];
  ρd2 = PP_ExpandedMold [
    Expand [F[{}, {}] × Normal@Series [ρd1, {e, 0, d}] // F[fs_, {es___}] ×
      (f : (r | γ)_{ps_} [is_])^{p-} => F[Join [fs, Table [f, p]], DeleteDuplicates@{es, is}]
    ];
  ρd3 = PP_pands@Expand [
    ρd2 /. F[fs_, es_] => Expand [gPair [
      Replace [fs, Thread [es → Range@Length@es], {2}],
      Length@es
    ] /. g_{α,β} => G[[es[[α]], es[[β]]]
  ];
  PP_Factor@Collect [Expand@{Δ, ρd3 /. e^{p-} → p! Δ^{2p} e^p} /. {T^{p-} /; p > 0 =>
    (
      (2 + z^2 - z √(4 + z^2)) / 2
    )^p, T^{p-} /; p < 0 => (
      (2 + z^2 + z √(4 + z^2)) / 2
    )^{-p}}, e, Expand]
  ];

```

Testing

In[*]:= ρ1[Knot[3, 1]]

Out[*]=

{1 + z^2, 1 + (2 z^2 + z^4) ∈}

```
In[*]:= TableForm[Table[Join[{K[[1]]K[[2]]}, ρ1[K]], {K, AllKnots[{3, 6}]}, TableAlignments → Center]
```

Out[*]//TableForm=

3_1	$1 + z^2$	$1 + (2z^2 + z^4) \in$
4_1	$1 - z^2$	1
5_1	$1 + 3z^2 + z^4$	$1 + (10z^2 + 21z^4 + 12z^6 + 2z^8) \in$
5_2	$1 + 2z^2$	$1 + (6z^2 + 5z^4) \in$
6_1	$1 - 2z^2$	$1 + (-2z^2 + z^4) \in$
6_2	$1 - z^2 - z^4$	$1 + (-2z^2 - 3z^4 + 2z^6 + z^8) \in$
6_3	$1 + z^2 + z^4$	1

```
In[*]:= ρ2[Knot[3, 1]]
```

- Running gPair[{r_{2,-1}[1, 2]}, 2]...
- Running gPair[{γ_{2,-1}[1]}, 1]...
- Running gPair[{γ_{2,0}[1]}, 1]...
- Running gPair[{r_{1,-1}[1, 2], r_{1,-1}[1, 2]}, 2]...
- Running gPair[{r_{1,-1}[1, 2], r_{1,-1}[3, 4]}, 4]...
- Running gPair[{r_{1,-1}[1, 2], γ_{1,-1}[3]}, 3]...
- Running gPair[{r_{1,-1}[1, 2], γ_{1,0}[3]}, 3]...
- Running gPair[{r_{1,-1}[1, 2], γ_{1,0}[1]}, 2]...
- Running gPair[{r_{1,-1}[1, 2], γ_{1,0}[2]}, 2]...
- Running gPair[{r_{1,-1}[1, 2], γ_{1,-1}[1]}, 2]...
- Running gPair[{γ_{1,-1}[1], γ_{1,-1}[1]}, 1]...
- Running gPair[{γ_{1,-1}[1], γ_{1,0}[2]}, 2]...
- Running gPair[{γ_{1,0}[1], γ_{1,0}[1]}, 1]...
- Running gPair[{γ_{1,0}[1], γ_{1,0}[2]}, 2]...

Out[*]=

$$\{1 + z^2, 1 + (2z^2 + z^4) \in + (2 - 4z^2 + 3z^4 + 4z^6 + z^8) \in^2\}$$

```
In[ ]:= BeginProfile []
```

```
Timing[z1 =  $\rho_2$ [Knot[10, 106]]]
```

```
PrintProfile []
```

```
Out[ ]:=
```

```
ProfileRoot
```

```
Out[ ]:=
```

$$\left\{ 3.53125, \left\{ 1 - z^2 - 5z^4 - 4z^6 - z^8, 1 + \left(2z^2 + 19z^4 + 16z^6 - 20z^8 - 38z^{10} - 25z^{12} - 8z^{14} - z^{16} \right) \epsilon + \left(-2 - 36z^2 + 49z^4 + 308z^6 + 1627z^8 + 4688z^{10} + 6814z^{12} + 5292z^{14} + 2040z^{16} + 504z^{18} + 929z^{20} + 1368z^{22} + 1009z^{24} + 432z^{26} + 111z^{28} + 16z^{30} + z^{32} \right) \epsilon^2 \right\} \right\}$$

```
Out[ ]:=
```

```
ProfileRoot is root. Profiled time: 3.531
```

```
( 1) 0.016/ 3.530 above  $\rho d$ 
```

```
PandS: called 1 times, time in 1.782/1.782
```

```
( 1) 1.780/ 1.780 under  $\rho d$ 
```

```
Green: called 1 times, time in 1.297/1.297
```

```
( 1) 1.300/ 1.300 under  $\rho d$ 
```

```
Factor: called 1 times, time in 0.343/0.343
```

```
( 1) 0.343/ 0.343 under  $\rho d$ 
```

```
ExpandedMold: called 1 times, time in 0.093/0.093
```

```
( 1) 0.093/ 0.093 under  $\rho d$ 
```

```
 $\rho d$ : called 1 times, time in 0.016/3.531
```

```
( 1) 0.016/ 3.530 under ProfileRoot
```

```
( 1) 1.300/ 1.300 above Green
```

```
( 1) 0.093/ 0.093 above ExpandedMold
```

```
( 1) 0.343/ 0.343 above Factor
```

```
( 1) 0/ 0 above Mold
```

```
( 1) 1.780/ 1.780 above PandS
```

```
Mold: called 1 times, time in 0./0.
```

```
( 1) 0/ 0 under  $\rho d$ 
```

```
In[ ]:= BeginProfile []
```

```
Timing[z2 =  $\rho_2$ [Knot[12, NonAlternating, 369]]]
```

```
PrintProfile []
```

```
Out[ ]:=
```

```
ProfileRoot
```

```
KnotTheory: Loading precomputed data in KnotTheory/12N.dts.
```

```
KnotTheory: The GaussCode to PD conversion was written by Siddarth Sankaran at the University of Toronto in the summer of 2005.
```

```
Running gPair[{\gamma_{2,1}[1]},1]...
Running gPair[{\rho_{1,-1}[1,2], \gamma_{1,1}[3]},3]...
Running gPair[{\rho_{1,-1}[1,2], \gamma_{1,1}[2]},2]...
Running gPair[{\rho_{1,1}[1,2], \gamma_{1,1}[3]},3]...
Running gPair[{\rho_{1,1}[1,2], \gamma_{1,1}[1]},2]...
Running gPair[{\gamma_{1,-1}[1], \gamma_{1,1}[2]},2]...
Running gPair[{\gamma_{1,0}[1], \gamma_{1,1}[2]},2]...
Running gPair[{\gamma_{1,1}[1], \gamma_{1,1}[1]},1]...
Running gPair[{\gamma_{1,1}[1], \gamma_{1,1}[2]},2]...
```

Out[]:=

$$\left\{ 7.59375, \left\{ 1 - z^2 - 5z^4 - 4z^6 - z^8, 1 + \left(2z^2 + 19z^4 + 16z^6 - 20z^8 - 38z^{10} - 25z^{12} - 8z^{14} - z^{16} \right) \in + \right. \right. \\ \left. \left. \left(-2 - 36z^2 + 17z^4 + 252z^6 + 1739z^8 + 5152z^{10} + 7798z^{12} + 7220z^{14} + 4888z^{16} + \right. \right. \right. \\ \left. \left. \left. 3208z^{18} + 2513z^{20} + 1920z^{22} + 1113z^{24} + 440z^{26} + 111z^{28} + 16z^{30} + z^{32} \right) \in^2 \right\} \right\}$$

Out[]:=

```
ProfileRoot is root. Profiled time: 7.593
( 1) 0/ 7.593 above \rho d
PandS: called 1 times, time in 4.719/4.797
( 1) 4.719/ 4.797 under \rho d
( 9) 0.078/ 0.078 above gPair
Green: called 1 times, time in 2./2.
( 1) 2.000/ 2.000 under \rho d
Factor: called 1 times, time in 0.609/0.609
( 1) 0.609/ 0.609 under \rho d
ExpandedMold: called 1 times, time in 0.187/0.187
( 1) 0.187/ 0.187 under \rho d
gPair: called 9 times, time in 0.078/0.078
( 9) 0.078/ 0.078 under PandS
\rho d: called 1 times, time in 0./7.593
( 1) 0/ 7.593 under ProfileRoot
( 1) 2.000/ 2.000 above Green
( 1) 0.187/ 0.187 above ExpandedMold
( 1) 0.609/ 0.609 above Factor
( 1) 0/ 0 above Mold
( 1) 4.719/ 4.797 above PandS
Mold: called 1 times, time in 0./0.
( 1) 0/ 0 under \rho d
```

In[]:= **Simplify**[Thread[z1 == z2]]

Out[]:=

$$\{ \text{True}, z (1 + z^2) (-4 + 5z^2 + 11z^4 + 14z^6 + 43z^8 + 59z^{10} + 36z^{12} + 10z^{14} + z^{16}) \in == 0 \}$$

```
In[*]:= TableForm[Table[Join[{K[[1]]K[[2]]}, ρ2[K]], {K, AllKnots[{3, 7}]}], TableAlignments → Center]
```

Out[*]//TableForm=

3 ₁	$1 + z^2$		$1 + (2 z^2 +$
4 ₁	$1 - z^2$		
5 ₁	$1 + 3 z^2 + z^4$	$1 + (10 z^2 + 21 z^4 + 12 z^6 + 2 z^8) \in + (6 -$	
5 ₂	$1 + 2 z^2$		$1 + (6 z^2 + 5 z'$
6 ₁	$1 - 2 z^2$		$1 + (-2 z^2 + ;$
6 ₂	$1 - z^2 - z^4$	$1 + (-2 z^2 - 3 z^4 + 2 z^6 + z^8) \in + ($	
6 ₃	$1 + z^2 + z^4$		$1 + (2 + 8$
7 ₁	$1 + 6 z^2 + 5 z^4 + z^6$	$1 + (28 z^2 + 126 z^4 + 180 z^6 + 110 z^8 + 30 z^{10} + 3 z^{12}) \in + (12 - 104 z^2 + 186 z^4 + 5$	
7 ₂	$1 + 3 z^2$		$1 + (12 z^2 + 14 z^4$
7 ₃	$1 + 5 z^2 + 2 z^4$	$1 + (-22 z^2 - 65 z^4 - 46 z^6 - 9 z^8) \in + (10 - 92$	
7 ₄	$1 + 4 z^2$		$1 + (-16 z^2 - 24 z$
7 ₅	$1 + 4 z^2 + 2 z^4$	$1 + (16 z^2 + 46 z^4 + 38 z^6 + 9 z^8) \in + (8 - 56$	
7 ₆	$1 + z^2 - z^4$	$1 + (4 z^2 - 4 z^4 - 2 z^6 + z^8) \in + (z$	
7 ₇	$1 - z^2 + z^4$		$1 + (2 z^2 - 3 z^4) \in +$

```
In[*]:= GST48 = EPD[X14,1, X2,29, X3,40, X43,4, X26,5, X6,95, X96,7, X13,8, X9,28, X10,41, X42,11, X27,12,  
X30,15, X16,61, X17,72, X18,83, X19,34, X89,20, X21,92, X79,22, X68,23, X57,24, X25,56, X62,31,  
X73,32, X84,33, X50,35, X36,81, X37,70, X38,59, X39,54, X44,55, X58,45, X69,46, X80,47, X48,91,  
X90,49, X51,82, X52,71, X53,60, X63,74, X64,85, X76,65, X87,66, X67,94, X75,86, X88,77, X78,93];
```

```
BeginProfile []  
Timing[z3 = ρ2[GST48]]  
PrintProfile []
```

Out[*]=

ProfileRoot

Out[]:=

$$\left\{ 570.922, \left\{ -\frac{(-1 + 2T - T^2 - T^3 + 2T^4 - T^5 + T^8)(-1 + T^3 - 2T^4 + T^5 + T^6 - 2T^7 + T^8)}{T^8}, \right. \right.$$

$$1 + \frac{1}{T^{16}} (-1 + T)^2 (5 - 18T + 33T^2 - 32T^3 + 2T^4 + 42T^5 - 62T^6 - 8T^7 + 166T^8 - 242T^9 + 108T^{10} +$$

$$132T^{11} - 226T^{12} + 148T^{13} - 11T^{14} - 36T^{15} - 11T^{16} + 148T^{17} - 226T^{18} + 132T^{19} + 108T^{20} -$$

$$242T^{21} + 166T^{22} - 8T^{23} - 62T^{24} + 42T^{25} + 2T^{26} - 32T^{27} + 33T^{28} - 18T^{29} + 5T^{30}) \in +$$

$$\frac{1}{2T^{32}} (25 - 348T + 2312T^2 - 9628T^3 + 27228T^4 - 51460T^5 + 52250T^6 + 25828T^7 -$$

$$197145T^8 + 313268T^9 - 36579T^{10} - 887864T^{11} + 2118398T^{12} - 2494152T^{13} + 772387T^{14} +$$

$$2785204T^{15} - 5477089T^{16} + 3765568T^{17} + 2886710T^{18} - 9712796T^{19} + 9746285T^{20} -$$

$$708568T^{21} - 11443177T^{22} + 17013304T^{23} - 11217405T^{24} - 1334300T^{25} + 10332369T^{26} -$$

$$8571752T^{27} - 1186874T^{28} + 8007252T^{29} - 3568015T^{30} - 8148860T^{31} + 14395240T^{32} -$$

$$8148860T^{33} - 3568015T^{34} + 8007252T^{35} - 1186874T^{36} - 8571752T^{37} + 10332369T^{38} -$$

$$1334300T^{39} - 11217405T^{40} + 17013304T^{41} - 11443177T^{42} - 708568T^{43} + 9746285T^{44} -$$

$$9712796T^{45} + 2886710T^{46} + 3765568T^{47} - 5477089T^{48} + 2785204T^{49} + 772387T^{50} -$$

$$2494152T^{51} + 2118398T^{52} - 887864T^{53} - 36579T^{54} + 313268T^{55} - 197145T^{56} +$$

$$25828T^{57} + 52250T^{58} - 51460T^{59} + 27228T^{60} - 9628T^{61} + 2312T^{62} - 348T^{63} + 25T^{64}) \in^2 \left. \right\}$$

Out[]:=

```
ProfileRoot is root. Profiled time: 570.922
( 1) 0.172/ 570.922 above ρd
PandS: called 1 times, time in 374.844/374.844
( 1) 374.844/ 374.844 under ρd
Green: called 1 times, time in 188.094/188.094
( 1) 188.094/ 188.094 under ρd
ExpandedMold: called 1 times, time in 7.75/7.75
( 1) 7.750/ 7.750 under ρd
ρd: called 1 times, time in 0.172/570.922
( 1) 0.172/ 570.922 under ProfileRoot
( 1) 188.094/ 188.094 above Green
( 1) 7.750/ 7.750 above ExpandedMold
( 1) 0.062/ 0.062 above Factor
( 1) 0/ 0 above Mold
( 1) 374.844/ 374.844 above PandS
Factor: called 1 times, time in 0.062/0.062
( 1) 0.062/ 0.062 under ρd
Mold: called 1 times, time in 0./0.
( 1) 0/ 0 under ρd
```

```
In[ ]:= BeginProfile[]
Timing[ρ₃[Knot[3, 1]]]
PrintProfile[]
```

Out[]:=

```
ProfileRoot
Running gPair[{r₃,-1[1, 2]}, 2]...
```



```

Running gPair[{\gamma_{3,-1}[1]},1]...
Running gPair[{\gamma_{3,\emptyset}[1]},1]...
Running gPair[{r_{1,-1}[1,2],r_{2,-1}[1,2]},2]...
Running gPair[{r_{1,-1}[1,2],r_{2,-1}[3,4]},4]...
Running gPair[{r_{1,-1}[1,2],\gamma_{2,-1}[3]},3]...
Running gPair[{r_{1,-1}[1,2],\gamma_{2,\emptyset}[3]},3]...
Running gPair[{r_{1,-1}[1,2],\gamma_{2,\emptyset}[1]},2]...
Running gPair[{r_{1,-1}[1,2],\gamma_{2,\emptyset}[2]},2]...
Running gPair[{r_{1,-1}[1,2],\gamma_{2,-1}[1]},2]...
Running gPair[{r_{2,-1}[1,2],\gamma_{1,-1}[3]},3]...
Running gPair[{r_{2,-1}[1,2],\gamma_{1,\emptyset}[3]},3]...
Running gPair[{r_{2,-1}[1,2],\gamma_{1,\emptyset}[1]},2]...
Running gPair[{r_{2,-1}[1,2],\gamma_{1,\emptyset}[2]},2]...
Running gPair[{r_{2,-1}[1,2],\gamma_{1,-1}[1]},2]...
Running gPair[{\gamma_{1,-1}[1],\gamma_{2,-1}[1]},1]...
Running gPair[{\gamma_{1,-1}[1],\gamma_{2,\emptyset}[2]},2]...
Running gPair[{\gamma_{1,\emptyset}[1],\gamma_{2,-1}[2]},2]...
Running gPair[{\gamma_{1,\emptyset}[1],\gamma_{2,\emptyset}[1]},1]...
Running gPair[{\gamma_{1,\emptyset}[1],\gamma_{2,\emptyset}[2]},2]...
Running gPair[{r_{1,-1}[1,2],r_{1,-1}[1,2],r_{1,-1}[1,2]},2]...
Running gPair[{r_{1,-1}[1,2],r_{1,-1}[1,2],r_{1,-1}[3,4]},4]...
Running gPair[{r_{1,-1}[1,2],r_{1,-1}[1,2],\gamma_{1,-1}[3]},3]...
Running gPair[{r_{1,-1}[1,2],r_{1,-1}[1,2],\gamma_{1,\emptyset}[3]},3]...
Running gPair[{r_{1,-1}[1,2],r_{1,-1}[1,2],\gamma_{1,\emptyset}[1]},2]...
Running gPair[{r_{1,-1}[1,2],r_{1,-1}[1,2],\gamma_{1,\emptyset}[2]},2]...
Running gPair[{r_{1,-1}[1,2],r_{1,-1}[3,4],r_{1,-1}[3,4]},4]...
Running gPair[{r_{1,-1}[1,2],r_{1,-1}[3,4],r_{1,-1}[5,6]},6]...
Running gPair[{r_{1,-1}[1,2],r_{1,-1}[3,4],\gamma_{1,-1}[3]},4]...
Running gPair[{r_{1,-1}[1,2],r_{1,-1}[3,4],\gamma_{1,\emptyset}[4]},4]...
Running gPair[{r_{1,-1}[1,2],r_{1,-1}[3,4],\gamma_{1,\emptyset}[1]},4]...
Running gPair[{r_{1,-1}[1,2],r_{1,-1}[3,4],\gamma_{1,\emptyset}[5]},5]...
Running gPair[{r_{1,-1}[1,2],r_{1,-1}[3,4],\gamma_{1,\emptyset}[2]},4]...
Running gPair[{r_{1,-1}[1,2],r_{1,-1}[3,4],\gamma_{1,-1}[5]},5]...
Running gPair[{r_{1,-1}[1,2],r_{1,-1}[3,4],\gamma_{1,\emptyset}[3]},4]...
Running gPair[{r_{1,-1}[1,2],\gamma_{1,-1}[3],\gamma_{1,-1}[3]},3]...
Running gPair[{r_{1,-1}[1,2],\gamma_{1,-1}[3],\gamma_{1,\emptyset}[4]},4]...

```

```

Running gPair[{r1,-1[1, 2], γ1,-1[3], γ1,θ[1]},3]...
Running gPair[{r1,-1[1, 2], γ1,-1[3], γ1,θ[2]},3]...
Running gPair[{r1,-1[1, 2], γ1,θ[3], γ1,θ[3]},3]...
Running gPair[{r1,-1[1, 2], γ1,θ[3], γ1,θ[1]},3]...
Running gPair[{r1,-1[1, 2], γ1,θ[3], γ1,θ[4]},4]...
Running gPair[{r1,-1[1, 2], γ1,θ[3], γ1,θ[2]},3]...
Running gPair[{r1,-1[1, 2], γ1,θ[1], γ1,θ[1]},2]...
Running gPair[{r1,-1[1, 2], γ1,θ[1], γ1,θ[3]},3]...
Running gPair[{r1,-1[1, 2], γ1,θ[1], γ1,θ[2]},2]...
Running gPair[{r1,-1[1, 2], γ1,θ[2], γ1,θ[2]},2]...
Running gPair[{r1,-1[1, 2], γ1,θ[2], γ1,θ[3]},3]...
Running gPair[{r1,-1[1, 2], r1,-1[1, 2], γ1,-1[1]},2]...
Running gPair[{r1,-1[1, 2], r1,-1[3, 4], γ1,-1[1]},4]...
Running gPair[{r1,-1[1, 2], γ1,-1[1], γ1,-1[1]},2]...
Running gPair[{r1,-1[1, 2], γ1,-1[1], γ1,θ[2]},2]...
Running gPair[{r1,-1[1, 2], γ1,-1[1], γ1,θ[3]},3]...
Running gPair[{r1,-1[1, 2], γ1,θ[2], γ1,θ[1]},2]...
Running gPair[{γ1,-1[1], γ1,-1[1], γ1,-1[1]},1]...
Running gPair[{γ1,-1[1], γ1,-1[1], γ1,θ[2]},2]...
Running gPair[{γ1,-1[1], γ1,θ[2], γ1,θ[2]},2]...
Running gPair[{γ1,-1[1], γ1,θ[2], γ1,θ[3]},3]...
Running gPair[{γ1,θ[1], γ1,θ[1], γ1,θ[1]},1]...
Running gPair[{γ1,θ[1], γ1,θ[1], γ1,θ[2]},2]...
Running gPair[{γ1,θ[1], γ1,θ[2], γ1,θ[2]},2]...
Running gPair[{γ1,θ[1], γ1,θ[2], γ1,θ[3]},3]...

```

Out[*]=

$$\left\{ 36.3438, \frac{\left(\frac{1 - T + T^2}{T}, 1 + \frac{(-1 + T)^2 (1 + T^2)}{T^2} + \frac{(1 - 4T + 7T^2 - 12T^3 + 18T^4 - 12T^5 + 7T^6 - 4T^7 + T^8) \epsilon^2}{2T^4} + \frac{(1 - 6T + 14T^2 - 34T^3 + 92T^4 - 98T^5 + 50T^6 - 98T^7 + 92T^8 - 34T^9 + 14T^{10} - 6T^{11} + T^{12}) \epsilon^3}{6T^6} \right) \right\}$$

Out[]:=

```

ProfileRoot is root. Profiled time: 36.344
( 1)      0/ 36.344 above ρd
gPair: called 62 times, time in 35.126/35.126
( 62) 35.126/ 35.126 under PandS
PandS: called 1 times, time in 1.124/36.25
( 1)  1.124/ 36.250 under ρd
( 62) 35.126/ 35.126 above gPair
ExpandedMold: called 1 times, time in 0.062/0.062
( 1)  0.062/  0.062 under ρd
Factor: called 1 times, time in 0.016/0.016
( 1)  0.016/  0.016 under ρd
Green: called 1 times, time in 0.016/0.016
( 1)  0.016/  0.016 under ρd
ρd: called 1 times, time in 0./36.344
( 1)      0/ 36.344 under ProfileRoot
( 1)  0.016/  0.016 above Green
( 1)  0.062/  0.062 above ExpandedMold
( 1)  0.016/  0.016 above Factor
( 1)      0/      0 above Mold
( 1)  1.124/ 36.250 above PandS
Mold: called 1 times, time in 0./0.
( 1)      0/      0 under ρd

```

In[]:=

```

BeginProfile[]
Timing[ρ3[Knot[4, 1]]]
PrintProfile[]

```

Out[]:=

```

ProfileRoot
Running gPair[{r3,1[1, 2]}, 2]...
Running gPair[{r1,-1[1, 2], r2,1[3, 4]}, 4]...
Running gPair[{r1,1[1, 2], r2,-1[3, 4]}, 4]...
Running gPair[{r1,1[1, 2], r2,1[1, 2]}, 2]...
Running gPair[{r1,1[1, 2], r2,1[3, 4]}, 4]...
Running gPair[{r1,1[1, 2], γ2,-1[2]}, 2]...
Running gPair[{r1,1[1, 2], γ2,-1[3]}, 3]...
Running gPair[{r1,1[1, 2], γ2,0[1]}, 2]...
Running gPair[{r1,1[1, 2], γ2,0[3]}, 3]...
Running gPair[{r1,1[1, 2], γ2,0[2]}, 2]...
Running gPair[{r2,1[1, 2], γ1,-1[2]}, 2]...
Running gPair[{r2,1[1, 2], γ1,-1[3]}, 3]...
Running gPair[{r2,1[1, 2], γ1,0[1]}, 2]...
Running gPair[{r2,1[1, 2], γ1,0[3]}, 3]...

```

```

Running gPair[{r2,1[1, 2],  $\gamma_{1,0}[2]$ },2]...
Running gPair[{\mathit{\gamma}_{1,-1}[1],  $\gamma_{2,-1}[2]$ },2]...
Running gPair[{r1,-1[1, 2], r1,-1[1, 2], r1,1[3, 4]},4]...
Running gPair[{r1,-1[1, 2], r1,-1[3, 4], r1,1[5, 6]},6]...
Running gPair[{r1,-1[1, 2], r1,1[3, 4], r1,1[3, 4]},4]...
Running gPair[{r1,-1[1, 2], r1,1[3, 4], r1,1[5, 6]},6]...
Running gPair[{r1,-1[1, 2], r1,1[3, 4],  $\gamma_{1,-1}[4]$ },4]...
Running gPair[{r1,-1[1, 2], r1,1[3, 4],  $\gamma_{1,-1}[5]$ },5]...
Running gPair[{r1,-1[1, 2], r1,1[3, 4],  $\gamma_{1,0}[3]$ },4]...
Running gPair[{r1,-1[1, 2], r1,1[3, 4],  $\gamma_{1,0}[5]$ },5]...
Running gPair[{r1,-1[1, 2], r1,1[3, 4],  $\gamma_{1,0}[1]$ },4]...
Running gPair[{r1,-1[1, 2], r1,1[3, 4],  $\gamma_{1,0}[2]$ },4]...
Running gPair[{r1,-1[1, 2], r1,1[3, 4],  $\gamma_{1,0}[4]$ },4]...
Running gPair[{r1,-1[1, 2],  $\gamma_{1,-1}[3]$ ,  $\gamma_{1,-1}[4]$ },4]...
Running gPair[{r1,-1[1, 2], r1,1[3, 4],  $\gamma_{1,-1}[1]$ },4]...
Running gPair[{r1,-1[1, 2],  $\gamma_{1,-1}[3]$ ,  $\gamma_{1,-1}[1]$ },3]...
Running gPair[{r1,1[1, 2], r1,1[1, 2], r1,1[1, 2]},2]...
Running gPair[{r1,1[1, 2], r1,1[1, 2], r1,1[3, 4]},4]...
Running gPair[{r1,1[1, 2], r1,1[1, 2],  $\gamma_{1,-1}[2]$ },2]...
Running gPair[{r1,1[1, 2], r1,1[1, 2],  $\gamma_{1,-1}[3]$ },3]...
Running gPair[{r1,1[1, 2], r1,1[1, 2],  $\gamma_{1,0}[1]$ },2]...
Running gPair[{r1,1[1, 2], r1,1[1, 2],  $\gamma_{1,0}[3]$ },3]...
Running gPair[{r1,1[1, 2], r1,1[3, 4], r1,1[3, 4]},4]...
Running gPair[{r1,1[1, 2], r1,1[3, 4],  $\gamma_{1,-1}[2]$ },4]...
Running gPair[{r1,1[1, 2], r1,1[3, 4],  $\gamma_{1,-1}[5]$ },5]...
Running gPair[{r1,1[1, 2], r1,1[3, 4],  $\gamma_{1,0}[1]$ },4]...
Running gPair[{r1,1[1, 2], r1,1[3, 4],  $\gamma_{1,0}[5]$ },5]...
Running gPair[{r1,1[1, 2], r1,1[3, 4],  $\gamma_{1,0}[3]$ },4]...
Running gPair[{r1,1[1, 2], r1,1[3, 4],  $\gamma_{1,0}[4]$ },4]...
Running gPair[{r1,1[1, 2],  $\gamma_{1,-1}[2]$ ,  $\gamma_{1,-1}[2]$ },2]...
Running gPair[{r1,1[1, 2],  $\gamma_{1,-1}[2]$ ,  $\gamma_{1,-1}[3]$ },3]...
Running gPair[{r1,1[1, 2],  $\gamma_{1,-1}[2]$ ,  $\gamma_{1,0}[1]$ },2]...
Running gPair[{r1,1[1, 2],  $\gamma_{1,-1}[2]$ ,  $\gamma_{1,0}[3]$ },3]...
Running gPair[{r1,1[1, 2],  $\gamma_{1,-1}[3]$ ,  $\gamma_{1,-1}[3]$ },3]...
Running gPair[{r1,1[1, 2],  $\gamma_{1,-1}[3]$ ,  $\gamma_{1,0}[1]$ },3]...
Running gPair[{r1,1[1, 2],  $\gamma_{1,-1}[3]$ ,  $\gamma_{1,0}[4]$ },4]...

```

```

Running gPair[{r1,1[1, 2], γ1,0[1], γ1,0[1]},2]...
Running gPair[{r1,1[1, 2], γ1,0[1], γ1,0[3]},3]...
Running gPair[{r1,1[1, 2], γ1,0[3], γ1,0[3]},3]...
Running gPair[{r1,1[1, 2], γ1,0[3], γ1,0[4]},4]...
Running gPair[{r1,1[1, 2], r1,1[1, 2], γ1,0[2]},2]...
Running gPair[{r1,1[1, 2], γ1,-1[3], γ1,-1[4]},4]...
Running gPair[{r1,1[1, 2], γ1,-1[3], γ1,0[2]},3]...
Running gPair[{r1,1[1, 2], γ1,0[3], γ1,0[1]},3]...
Running gPair[{r1,1[1, 2], γ1,0[3], γ1,0[2]},3]...
Running gPair[{r1,1[1, 2], γ1,0[1], γ1,0[2]},2]...
Running gPair[{r1,1[1, 2], γ1,0[2], γ1,0[2]},2]...
Running gPair[{γ1,-1[1], γ1,-1[1], γ1,-1[2]},2]...
Running gPair[{γ1,-1[1], γ1,-1[2], γ1,-1[2]},2]...
Running gPair[{γ1,-1[1], γ1,-1[2], γ1,0[3]},3]...

```

Out[]=

$$\left\{ 73.4531, \left\{ -\frac{1 - 3T + T^2}{T}, 1 + \frac{(1 - 3T + T^2)(1 - T + T^2)\epsilon^2}{T^2} \right\} \right\}$$

Out[]=

```

ProfileRoot is root. Profiled time: 73.453
( 1) 0/ 73.453 above ρd
gPair: called 64 times, time in 71.611/71.611
( 64) 71.611/ 71.611 under PandS
PandS: called 1 times, time in 1.67/73.281
( 1) 1.670/ 73.281 under ρd
( 64) 71.611/ 71.611 above gPair
ExpandedMold: called 1 times, time in 0.14/0.14
( 1) 0.140/ 0.140 under ρd
Factor: called 1 times, time in 0.016/0.016
( 1) 0.016/ 0.016 under ρd
Green: called 1 times, time in 0.016/0.016
( 1) 0.016/ 0.016 under ρd
ρd: called 1 times, time in 0./73.453
( 1) 0/ 73.453 under ProfileRoot
( 1) 0.016/ 0.016 above Green
( 1) 0.140/ 0.140 above ExpandedMold
( 1) 0.016/ 0.016 above Factor
( 1) 0/ 0 above Mold
( 1) 1.670/ 73.281 above PandS
Mold: called 1 times, time in 0./0.
( 1) 0/ 0 under ρd

```

```
In[ ]:= TableForm[Table[Echo@Join[{K[[1]]K[[2]]}, ρ3[K]], {K, AllKnots[{3, 6}]}],
  TableAlignments → Center]
```

Running gPair[{r_{3,-1}[1, 2]}, 2]...

Running gPair[{γ_{3,-1}[1]}, 1]...

Running gPair[{γ_{3,0}[1]}, 1]...

Running gPair[{r_{1,-1}[1, 2], r_{2,-1}[1, 2]}, 2]...

Running gPair[{r_{1,-1}[1, 2], r_{2,-1}[3, 4]}, 4]...

Running gPair[{r_{1,-1}[1, 2], γ_{2,-1}[3]}, 3]...

Running gPair[{r_{1,-1}[1, 2], γ_{2,0}[3]}, 3]...

Running gPair[{r_{1,-1}[1, 2], γ_{2,0}[1]}, 2]...

Running gPair[{r_{1,-1}[1, 2], γ_{2,0}[2]}, 2]...

Running gPair[{r_{1,-1}[1, 2], γ_{2,-1}[1]}, 2]...

Running gPair[{r_{2,-1}[1, 2], γ_{1,-1}[3]}, 3]...

Running gPair[{r_{2,-1}[1, 2], γ_{1,0}[3]}, 3]...

Running gPair[{r_{2,-1}[1, 2], γ_{1,0}[1]}, 2]...

Running gPair[{r_{2,-1}[1, 2], γ_{1,0}[2]}, 2]...

Running gPair[{r_{2,-1}[1, 2], γ_{1,-1}[1]}, 2]...

Running gPair[{γ_{1,-1}[1], γ_{2,-1}[1]}, 1]...

Running gPair[{γ_{1,-1}[1], γ_{2,0}[2]}, 2]...

Running gPair[{γ_{1,0}[1], γ_{2,-1}[2]}, 2]...

Running gPair[{γ_{1,0}[1], γ_{2,0}[1]}, 1]...

Running gPair[{γ_{1,0}[1], γ_{2,0}[2]}, 2]...

Running gPair[{r_{1,-1}[1, 2], r_{1,-1}[1, 2], r_{1,-1}[1, 2]}, 2]...

Running gPair[{r_{1,-1}[1, 2], r_{1,-1}[1, 2], r_{1,-1}[3, 4]}, 4]...

Running gPair[{r_{1,-1}[1, 2], r_{1,-1}[1, 2], γ_{1,-1}[3]}, 3]...

Running gPair[{r_{1,-1}[1, 2], r_{1,-1}[1, 2], γ_{1,0}[3]}, 3]...

Running gPair[{r_{1,-1}[1, 2], r_{1,-1}[1, 2], γ_{1,0}[1]}, 2]...

Running gPair[{r_{1,-1}[1, 2], r_{1,-1}[1, 2], γ_{1,0}[2]}, 2]...

Running gPair[{r_{1,-1}[1, 2], r_{1,-1}[3, 4], r_{1,-1}[3, 4]}, 4]...

Running gPair[{r_{1,-1}[1, 2], r_{1,-1}[3, 4], r_{1,-1}[5, 6]}, 6]...

Running gPair[{r_{1,-1}[1, 2], r_{1,-1}[3, 4], γ_{1,-1}[3]}, 4]...

Running gPair[{r_{1,-1}[1, 2], r_{1,-1}[3, 4], γ_{1,0}[4]}, 4]...

Running gPair[{r_{1,-1}[1, 2], r_{1,-1}[3, 4], γ_{1,0}[1]}, 4]...

Running gPair[{r_{1,-1}[1, 2], r_{1,-1}[3, 4], γ_{1,0}[5]}, 5]...

Running gPair[{r_{1,-1}[1, 2], r_{1,-1}[3, 4], γ_{1,0}[2]}, 4]...

Running gPair[{r_{1,-1}[1, 2], r_{1,-1}[3, 4], γ_{1,-1}[5]}, 5]...

```

Running gPair[{r1,-1[1, 2], r1,-1[3, 4], γ1,0[3]},4]...
Running gPair[{r1,-1[1, 2], γ1,-1[3], γ1,-1[3]},3]...
Running gPair[{r1,-1[1, 2], γ1,-1[3], γ1,0[4]},4]...
Running gPair[{r1,-1[1, 2], γ1,-1[3], γ1,0[1]},3]...
Running gPair[{r1,-1[1, 2], γ1,-1[3], γ1,0[2]},3]...
Running gPair[{r1,-1[1, 2], γ1,0[3], γ1,0[3]},3]...
Running gPair[{r1,-1[1, 2], γ1,0[3], γ1,0[1]},3]...
Running gPair[{r1,-1[1, 2], γ1,0[3], γ1,0[4]},4]...
Running gPair[{r1,-1[1, 2], γ1,0[3], γ1,0[2]},3]...
Running gPair[{r1,-1[1, 2], γ1,0[1], γ1,0[1]},2]...
Running gPair[{r1,-1[1, 2], γ1,0[1], γ1,0[3]},3]...
Running gPair[{r1,-1[1, 2], γ1,0[1], γ1,0[2]},2]...
Running gPair[{r1,-1[1, 2], γ1,0[2], γ1,0[2]},2]...
Running gPair[{r1,-1[1, 2], γ1,0[2], γ1,0[3]},3]...
Running gPair[{r1,-1[1, 2], r1,-1[1, 2], γ1,-1[1]},2]...
Running gPair[{r1,-1[1, 2], r1,-1[3, 4], γ1,-1[1]},4]...
Running gPair[{r1,-1[1, 2], γ1,-1[1], γ1,-1[1]},2]...
Running gPair[{r1,-1[1, 2], γ1,-1[1], γ1,0[2]},2]...
Running gPair[{r1,-1[1, 2], γ1,-1[1], γ1,0[3]},3]...
Running gPair[{r1,-1[1, 2], γ1,0[2], γ1,0[1]},2]...
Running gPair[{γ1,-1[1], γ1,-1[1], γ1,-1[1]},1]...
Running gPair[{γ1,-1[1], γ1,-1[1], γ1,0[2]},2]...
Running gPair[{γ1,-1[1], γ1,0[2], γ1,0[2]},2]...
Running gPair[{γ1,-1[1], γ1,0[2], γ1,0[3]},3]...
Running gPair[{γ1,0[1], γ1,0[1], γ1,0[1]},1]...
Running gPair[{γ1,0[1], γ1,0[1], γ1,0[2]},2]...
Running gPair[{γ1,0[1], γ1,0[2], γ1,0[2]},2]...
Running gPair[{γ1,0[1], γ1,0[2], γ1,0[3]},3]...
Running gPair[{r3,1[1, 2]},2]...
Running gPair[{r1,-1[1, 2], r2,1[3, 4]},4]...
Running gPair[{r1,1[1, 2], r2,-1[3, 4]},4]...
Running gPair[{r1,1[1, 2], r2,1[1, 2]},2]...
Running gPair[{r1,1[1, 2], r2,1[3, 4]},4]...
Running gPair[{r1,1[1, 2], γ2,-1[2]},2]...
Running gPair[{r1,1[1, 2], γ2,-1[3]},3]...

```

$\gg \{3_1, 1 + z^2, 1 + (2z^2 + z^4) \in + (2 - 4z^2 + 3z^4 + 4z^6 + z^8) \in^2 + (-12 + 74z^2 - 27z^4 - 20z^6 + 8z^8 + 6z^{10} + z^{12}) \in^3\}$

```

Running gPair[{r1,1[1, 2], γ2,0[1]},2]...
Running gPair[{r1,1[1, 2], γ2,0[3]},3]...
Running gPair[{r1,1[1, 2], γ2,0[2]},2]...
Running gPair[{r2,1[1, 2], γ1,-1[2]},2]...
Running gPair[{r2,1[1, 2], γ1,-1[3]},3]...
Running gPair[{r2,1[1, 2], γ1,0[1]},2]...
Running gPair[{r2,1[1, 2], γ1,0[3]},3]...
Running gPair[{r2,1[1, 2], γ1,0[2]},2]...
Running gPair[{γ1,-1[1], γ2,-1[2]},2]...
Running gPair[{r1,-1[1, 2], r1,-1[1, 2], r1,1[3, 4]},4]...
Running gPair[{r1,-1[1, 2], r1,-1[3, 4], r1,1[5, 6]},6]...
Running gPair[{r1,-1[1, 2], r1,1[3, 4], r1,1[3, 4]},4]...
Running gPair[{r1,-1[1, 2], r1,1[3, 4], r1,1[5, 6]},6]...
Running gPair[{r1,-1[1, 2], r1,1[3, 4], γ1,-1[4]},4]...
Running gPair[{r1,-1[1, 2], r1,1[3, 4], γ1,-1[5]},5]...
Running gPair[{r1,-1[1, 2], r1,1[3, 4], γ1,0[3]},4]...
Running gPair[{r1,-1[1, 2], r1,1[3, 4], γ1,0[5]},5]...
Running gPair[{r1,-1[1, 2], r1,1[3, 4], γ1,0[1]},4]...
Running gPair[{r1,-1[1, 2], r1,1[3, 4], γ1,0[2]},4]...
Running gPair[{r1,-1[1, 2], r1,1[3, 4], γ1,0[4]},4]...
Running gPair[{r1,-1[1, 2], γ1,-1[3], γ1,-1[4]},4]...
Running gPair[{r1,-1[1, 2], r1,1[3, 4], γ1,-1[1]},4]...
Running gPair[{r1,-1[1, 2], γ1,-1[3], γ1,-1[1]},3]...
Running gPair[{r1,1[1, 2], r1,1[1, 2], r1,1[1, 2]},2]...
Running gPair[{r1,1[1, 2], r1,1[1, 2], r1,1[3, 4]},4]...
Running gPair[{r1,1[1, 2], r1,1[1, 2], γ1,-1[2]},2]...
Running gPair[{r1,1[1, 2], r1,1[1, 2], γ1,-1[3]},3]...
Running gPair[{r1,1[1, 2], r1,1[1, 2], γ1,0[1]},2]...
Running gPair[{r1,1[1, 2], r1,1[1, 2], γ1,0[3]},3]...
Running gPair[{r1,1[1, 2], r1,1[3, 4], r1,1[3, 4]},4]...
Running gPair[{r1,1[1, 2], r1,1[3, 4], γ1,-1[2]},4]...
Running gPair[{r1,1[1, 2], r1,1[3, 4], γ1,-1[5]},5]...
Running gPair[{r1,1[1, 2], r1,1[3, 4], γ1,0[1]},4]...
Running gPair[{r1,1[1, 2], r1,1[3, 4], γ1,0[5]},5]...
Running gPair[{r1,1[1, 2], r1,1[3, 4], γ1,0[3]},4]...
Running gPair[{r1,1[1, 2], r1,1[3, 4], γ1,0[4]},4]...

```



```

Running gPair[{r1,1[1, 2], γ1,-1[2], γ1,-1[2]},2]...
Running gPair[{r1,1[1, 2], γ1,-1[2], γ1,-1[3]},3]...
Running gPair[{r1,1[1, 2], γ1,-1[2], γ1,0[1]},2]...
Running gPair[{r1,1[1, 2], γ1,-1[2], γ1,0[3]},3]...
Running gPair[{r1,1[1, 2], γ1,-1[3], γ1,-1[3]},3]...
Running gPair[{r1,1[1, 2], γ1,-1[3], γ1,0[1]},3]...
Running gPair[{r1,1[1, 2], γ1,-1[3], γ1,0[4]},4]...
Running gPair[{r1,1[1, 2], γ1,0[1], γ1,0[1]},2]...
Running gPair[{r1,1[1, 2], γ1,0[1], γ1,0[3]},3]...
Running gPair[{r1,1[1, 2], γ1,0[3], γ1,0[3]},3]...
Running gPair[{r1,1[1, 2], γ1,0[3], γ1,0[4]},4]...
Running gPair[{r1,1[1, 2], r1,1[1, 2], γ1,0[2]},2]...
Running gPair[{r1,1[1, 2], γ1,-1[3], γ1,-1[4]},4]...
Running gPair[{r1,1[1, 2], γ1,-1[3], γ1,0[2]},3]...
Running gPair[{r1,1[1, 2], γ1,0[3], γ1,0[1]},3]...
Running gPair[{r1,1[1, 2], γ1,0[3], γ1,0[2]},3]...
Running gPair[{r1,1[1, 2], γ1,0[1], γ1,0[2]},2]...
Running gPair[{r1,1[1, 2], γ1,0[2], γ1,0[2]},2]...
Running gPair[{γ1,-1[1], γ1,-1[1], γ1,-1[2]},2]...
Running gPair[{γ1,-1[1], γ1,-1[2], γ1,-1[2]},2]...
Running gPair[{γ1,-1[1], γ1,-1[2], γ1,0[3]},3]...
» {41, 1 - z2, 1 + (-2 + 2 z4) ε2}
» {51, 1 + 3 z2 + z4,
    1 + (10 z2 + 21 z4 + 12 z6 + 2 z8) ε + (6 - 28 z2 + 33 z4 + 364 z6 + 655 z8 + 536 z10 + 227 z12 + 48 z14 + 4 z16) ε2 +
    (-60 + 970 z2 + 645 z4 - 3380 z6 - 3280 z8 + 7470 z10 + 19475 z12 +
    20536 z14 + 12564 z16 + 4774 z18 + 1109 z20 + 144 z22 + 8 z24) ε3}
Running gPair[{γ3,1[1]},1]...
Running gPair[{r1,-1[1, 2], γ2,1[3]},3]...
Running gPair[{r1,-1[1, 2], γ2,1[2]},2]...
Running gPair[{r2,-1[1, 2], γ1,1[3]},3]...
Running gPair[{r2,-1[1, 2], γ1,1[2]},2]...
Running gPair[{γ1,-1[1], γ2,1[2]},2]...
Running gPair[{γ1,0[1], γ2,1[2]},2]...
Running gPair[{γ1,1[1], γ2,-1[2]},2]...
Running gPair[{γ1,1[1], γ2,0[2]},2]...
Running gPair[{γ1,1[1], γ2,1[1]},1]...

```

Running gPair[{r_{1,-1}[1, 2], r_{1,-1}[1, 2], $\gamma_{1,1}$ [3]}, 3] ...
 Running gPair[{r_{1,-1}[1, 2], r_{1,-1}[3, 4], $\gamma_{1,1}$ [5]}, 5] ...
 Running gPair[{r_{1,-1}[1, 2], r_{1,-1}[3, 4], $\gamma_{1,1}$ [4]}, 4] ...
 Running gPair[{r_{1,-1}[1, 2], $\gamma_{1,-1}$ [3], $\gamma_{1,1}$ [4]}, 4] ...
 Running gPair[{r_{1,-1}[1, 2], $\gamma_{1,0}$ [3], $\gamma_{1,1}$ [4]}, 4] ...
 Running gPair[{r_{1,-1}[1, 2], $\gamma_{1,0}$ [1], $\gamma_{1,1}$ [3]}, 3] ...
 Running gPair[{r_{1,-1}[1, 2], $\gamma_{1,0}$ [2], $\gamma_{1,1}$ [3]}, 3] ...
 Running gPair[{r_{1,-1}[1, 2], $\gamma_{1,1}$ [3], $\gamma_{1,1}$ [3]}, 3] ...
 Running gPair[{r_{1,-1}[1, 2], $\gamma_{1,-1}$ [1], $\gamma_{1,-1}$ [3]}, 3] ...
 Running gPair[{r_{1,-1}[1, 2], $\gamma_{1,-1}$ [1], $\gamma_{1,1}$ [3]}, 3] ...
 Running gPair[{r_{1,-1}[1, 2], r_{1,-1}[1, 2], $\gamma_{1,1}$ [2]}, 2] ...
 Running gPair[{r_{1,-1}[1, 2], r_{1,-1}[3, 4], $\gamma_{1,1}$ [2]}, 4] ...
 Running gPair[{r_{1,-1}[1, 2], $\gamma_{1,-1}$ [3], $\gamma_{1,1}$ [2]}, 3] ...
 Running gPair[{r_{1,-1}[1, 2], $\gamma_{1,0}$ [3], $\gamma_{1,1}$ [2]}, 3] ...
 Running gPair[{r_{1,-1}[1, 2], $\gamma_{1,0}$ [1], $\gamma_{1,1}$ [2]}, 2] ...
 Running gPair[{r_{1,-1}[1, 2], $\gamma_{1,1}$ [2], $\gamma_{1,1}$ [2]}, 2] ...
 Running gPair[{\mathit{\gamma}_{1,-1}[1], $\gamma_{1,-1}$ [1], $\gamma_{1,1}$ [2]}, 2] ...
 Running gPair[{\mathit{\gamma}_{1,-1}[1], $\gamma_{1,-1}$ [2], $\gamma_{1,1}$ [3]}, 3] ...
 Running gPair[{\mathit{\gamma}_{1,-1}[1], $\gamma_{1,0}$ [2], $\gamma_{1,1}$ [3]}, 3] ...
 Running gPair[{\mathit{\gamma}_{1,-1}[1], $\gamma_{1,1}$ [2], $\gamma_{1,1}$ [2]}, 2] ...
 Running gPair[{\mathit{\gamma}_{1,0}[1], $\gamma_{1,0}$ [1], $\gamma_{1,1}$ [2]}, 2] ...
 Running gPair[{\mathit{\gamma}_{1,0}[1], $\gamma_{1,0}$ [2], $\gamma_{1,1}$ [3]}, 3] ...
 Running gPair[{\mathit{\gamma}_{1,0}[1], $\gamma_{1,1}$ [2], $\gamma_{1,1}$ [2]}, 2] ...
 Running gPair[{\mathit{\gamma}_{1,1}[1], $\gamma_{1,1}$ [1], $\gamma_{1,1}$ [1]}, 1] ...
 » {5₂, 1 + 2 z², 1 + (6 z² + 5 z⁴) \in + (4 - 20 z² + 43 z⁴ + 64 z⁶ + 26 z⁸) \in^2 +
 (-36 + 498 z² - 883 z⁴ + 100 z⁶ + 816 z⁸ + 556 z¹⁰ + 146 z¹²) \in^3 }
 Running gPair[{r_{1,1}[1, 2], $\gamma_{2,1}$ [3]}, 3] ...
 Running gPair[{r_{2,1}[1, 2], $\gamma_{1,1}$ [3]}, 3] ...
 Running gPair[{r_{1,-1}[1, 2], r_{1,1}[3, 4], $\gamma_{1,1}$ [5]}, 5] ...
 Running gPair[{r_{1,-1}[1, 2], r_{1,1}[3, 4], $\gamma_{1,1}$ [2]}, 4] ...
 Running gPair[{r_{1,1}[1, 2], r_{1,1}[1, 2], $\gamma_{1,1}$ [3]}, 3] ...
 Running gPair[{r_{1,1}[1, 2], r_{1,1}[3, 4], $\gamma_{1,1}$ [5]}, 5] ...
 Running gPair[{r_{1,1}[1, 2], $\gamma_{1,-1}$ [3], $\gamma_{1,-1}$ [2]}, 3] ...
 Running gPair[{r_{1,1}[1, 2], $\gamma_{1,-1}$ [3], $\gamma_{1,1}$ [4]}, 4] ...
 Running gPair[{r_{1,1}[1, 2], $\gamma_{1,-1}$ [2], $\gamma_{1,1}$ [3]}, 3] ...
 Running gPair[{r_{1,1}[1, 2], $\gamma_{1,0}$ [3], $\gamma_{1,1}$ [4]}, 4] ...

Running gPair[{r_{1,1}[1, 2], γ_{1,0}[1], γ_{1,1}[3]}, 3]...

Running gPair[{r_{1,1}[1, 2], γ_{1,1}[3], γ_{1,1}[3]}, 3]...

Running gPair[{r_{1,1}[1, 2], γ_{1,0}[2], γ_{1,0}[3]}, 3]...

Running gPair[{r_{1,1}[1, 2], γ_{1,0}[2], γ_{1,0}[1]}, 2]...

Running gPair[{r_{1,1}[1, 2], γ_{1,0}[2], γ_{1,1}[3]}, 3]...

Running gPair[{γ_{1,-1}[1], γ_{1,-1}[2], γ_{1,-1}[3]}, 3]...

» {6₁, 1 - 2 z², 1 + (-2 z² + z⁴) ε + (-4 + 4 z² + 25 z⁴ - 8 z⁶ + 2 z⁸) ε² + (12 + 154 z² - 223 z⁴ - 608 z⁶ + 100 z⁸ - 52 z¹⁰ + 10 z¹²) ε³}

» {6₂, 1 - z² - z⁴, 1 + (-2 z² - 3 z⁴ + 2 z⁶ + z⁸) ε + (-2 - 4 z² + 29 z⁴ + 28 z⁶ + 42 z⁸ - 8 z¹⁰ - 2 z¹² + 4 z¹⁴ + z¹⁶) ε² + (12 + 166 z² + 155 z⁴ - 194 z⁶ - 2453 z⁸ - 1622 z¹⁰ - 1967 z¹² - 258 z¹⁴ + 49 z¹⁶ - 30 z¹⁸ + z²⁰ + 6 z²² + z²⁴) ε³}

Running gPair[{r_{1,1}[1, 2], r_{1,1}[3, 4], r_{1,1}[5, 6]}, 6]...

Running gPair[{r_{1,1}[1, 2], r_{1,1}[3, 4], γ_{1,0}[2]}, 4]...

» {6₃, 1 + z² + z⁴, 1 + (2 + 8 z² - 16 z⁶ - 24 z⁸ - 16 z¹⁰ - 2 z¹²) ε²}

Out[]//TableForm=

3 ₁	1 + z ²			1 + (2 z ² + z ⁴) ε
4 ₁	1 - z ²			
5 ₁	1 + 3 z ² + z ⁴	1 + (10 z ² + 21 z ⁴ + 12 z ⁶ + 2 z ⁸) ε +	(6 - 28 z ² + 33 z ⁴ + 364 z ⁶ + 655 z ⁸ + 536 z ¹⁰ + 227	
5 ₂	1 + 2 z ²		1 + (6 z ² + 5 z ⁴) ε +	(4 - 2
6 ₁	1 - 2 z ²		1 + (-2 z ² + z ⁴) ε +	(-
6 ₂	1 - z ² - z ⁴	1 + (-2 z ² - 3 z ⁴ + 2 z ⁶ + z ⁸) ε +	(-2 - 4 z ² + 29 z ⁴ + 28 z ⁶ + 42 z ⁸ - 8	
6 ₃	1 + z ² + z ⁴			

```

In[ ]:= BeginProfile []
Timing[z1 =  $\rho_3$ [Knot[11, NonAlternating, 34]]]
PrintProfile []

Out[ ]:=
ProfileRoot

KnotTheory: Loading precomputed data in DTCode4KnotsTo11`.

Out[ ]:=
$Aborted

Out[ ]:=
ProfileRoot is root. Profiled time: 4.687
( 1) 0/ 0 above  $\rho d$ 
ExpandedMold: called 1 times, time in 3.437/3.437
( 1) 3.440/ 3.440 under  $\rho d$ 
Green: called 1 times, time in 1.25/1.25
( 1) 1.250/ 1.250 under  $\rho d$ 
 $\rho d$ : called 1 times, time in 0./0.
( 1) 0/ 0 under ProfileRoot
( 1) 1.250/ 1.250 above Green
( 1) 3.440/ 3.440 above ExpandedMold
( 1) 0/ 0 above Mold
( 1) 0/ 0 above PandS
PandS: called 1 times, time in 0./0.
( 1) 0/ 0 under  $\rho d$ 
Mold: called 1 times, time in 0./0.
( 1) 0/ 0 under  $\rho d$ 

```

(Alt) In[]:=

```
BeginProfile []
Timing[z2 = ρ3[Knot[11, NonAlternating, 42]]]
PrintProfile []
```

(Alt) Out[]:=

ProfileRoot

(Alt) Out[]:=

$$\left\{ 196.234, \left\{ 1, 1 - \frac{2(-1+T)^2(1+T^4)}{T^3} + \frac{1}{T^6} 2(-1+T)^2(6-15T+12T^2+2T^3-3T^4-2T^5-3T^6+2T^7+12T^8-15T^9+6T^{10}) \epsilon^2 - \frac{1}{3T^9} \epsilon^3 (360-2520T+7632T^2-12510T^3+10899T^4-2988T^5-2942T^6+2731T^7-695T^8+54T^9-695T^{10}+2731T^{11}-2942T^{12}-2988T^{13}+10899T^{14}-12510T^{15}+7632T^{16}-2520T^{17}+360T^{18}-12T^6 ca_{3,1}+24T^7 ca_{3,1}-12T^8 ca_{3,1}-12T^{10} ca_{3,1}+24T^{11} ca_{3,1}-12T^{12} ca_{3,1}) \right\} \right\}$$

(Alt) Out[]:=

```
ProfileRoot is root. Profiled time: 196.235
( 1) 0/ 196.240 above ρd
PandS: called 1 times, time in 193.703/193.703
( 1) 193.700/ 193.700 under ρd
ExpandedMold: called 1 times, time in 2.016/2.016
( 1) 2.016/ 2.016 under ρd
Green: called 1 times, time in 0.516/0.516
( 1) 0.516/ 0.516 under ρd
ρd: called 1 times, time in 0./196.235
( 1) 0/ 196.240 under ProfileRoot
( 1) 0.516/ 0.516 above Green
( 1) 2.016/ 2.016 above ExpandedMold
( 1) 0/ 0 above Factor
( 1) 0/ 0 above Mold
( 1) 193.700/ 193.700 above PandS
Mold: called 1 times, time in 0./0.
( 1) 0/ 0 under ρd
Factor: called 1 times, time in 0./0.
( 1) 0/ 0 under ρd
```

(Alt) In[]:=

```
z1 - z2
```

(Alt) Out[]:=

{0, 0}

(Alt) In[]:=

```
BeginProfile []
Timing[ρ3[GST48]]
PrintProfile []
```

(Alt) Out[]:=

ProfileRoot

Running gPair[{r_{1,1}[1, 2], $\gamma_{2,1}$ [1]}, 2] ...

Running gPair[{r_{2,1}[1, 2], $\gamma_{1,1}$ [1]}, 2] ...

Running gPair[{r_{1,-1}[1, 2], r_{1,1}[3, 4], $\gamma_{1,1}$ [3]}, 4] ...