Pensieve header: Implementing ρ_1, and also ρ_d.

exec

```
nb2tex$TeXFileName = "Rho1.tex";
```

pdf

Preliminaries

pdf

This is Rho.nb of http://drorbn.net/oa22/ap.

In[•]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\Oaxaca-2210"];
```

pdf

In[•]:=
```
Once[<< KnotTheory`; << Rot.m];
```

pdf

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
```

pdf

```
Loading Rot.m from http://drorbn.net/la22/ap to compute rotation numbers.
```

pdf

The Program

pdf

In[•]:=
```
R1[s_, i_, j_] := s (gji (gj',j + gj,j' - gij) - gii (gj,j' - 1) - 1 / 2);
Z[K_] := Module[{Cs, φ, n, A, s, i, j, k, Δ, G, ρ1},
   {Cs, φ} = Rot[K]; n = Length[Cs];
   A = IdentityMatrix[2 n + 1];
   Cases[Cs, {s_, i_, j_} :> (A[[{i, j}, {i + 1, j + 1}]] += ((-T^s  T^s - 1
                                                               0    -1 )))];
   Δ = T^(-Total[φ] - Total[Cs[[All, 1]]]) / 2 Det[A];
   G = Inverse[A];
   ρ1 = Sum_{k=1}^{n} R1 @@ Cs[[k]] - Sum_{k=1}^{2 n} φ[[k]] (gkk - 1 / 2);
   Factor@{Δ, Δ^2 ρ1 /. α_^+ :> α + 1 /. gα_,β_ :> G[[α, β]]}];
```

pdf

The First Few Knots

pdf

In[ ]:=
```
TableForm[Table[Join[{K[[1]]_K[[2]]}, Z[K]], {K, AllKnots[{3, 6}]}], TableAlignments → Center]
```

pdf

KnotTheory: Loading precomputed data in PD4Knots`.

Out[ ]//TableForm=

pdf

3_1	$\frac{1-T+T^2}{T}$	$\frac{(-1+T)^2\,(1+T^2)}{T^2}$
4_1	$-\frac{1-3\,T+T^2}{T}$	0
5_1	$\frac{1-T+T^2-T^3+T^4}{T^2}$	$\frac{(-1+T)^2\,(1+T^2)\,(2+T^2+2\,T^4)}{T^4}$
5_2	$\frac{2-3\,T+2\,T^2}{T}$	$\frac{(-1+T)^2\,(5-4\,T+5\,T^2)}{T^2}$
6_1	$-\frac{(-2+T)\,(-1+2\,T)}{T}$	$\frac{(-1+T)^2\,(1-4\,T+T^2)}{T^2}$
6_2	$-\frac{1-3\,T+3\,T^2-3\,T^3+T^4}{T^2}$	$\frac{(-1+T)^2\,(1-4\,T+4\,T^2-4\,T^3+4\,T^4-4\,T^5+T^6)}{T^4}$
6_3	$\frac{1-3\,T+5\,T^2-3\,T^3+T^4}{T^2}$	0

tex

```
\def\nbpdfText#1{\vskip -3mm\[ \includegraphics[width=0.4\linewidth]{#1}\quad p=1-T^s \]}
```

pdf

tex

```
\def\nbpdfText#1{\vskip 1mm\par\noindent\includegraphics{#1}}
```

tex

```
\needspace{2in}
```

pdf

Fast!

tex

```
\[ \resizebox{\linewidth}{!}{\import{../Waco-2203/}{GST48-Marked.pdf_t}} \]
```

pdf

In[ ]:=
```
Timing@
 Z[GST48 = EPD[X14,1, X2,29, X3,40, X43,4, X26,5, X6,95, X96,7, X13,8, X9,28, X10,41, X42,11, X27,12, X30,15,
    X16,61, X17,72, X18,83, X19,34, X89,20, X21,92, X79,22, X68,23, X57,24, X25,56, X62,31, X73,32,
    X84,33, X50,35, X36,81, X37,70, X38,59, X39,54, X44,55, X58,45, X69,46, X80,47, X48,91, X90,49,
    X51,82, X52,71, X53,60, X63,74, X64,85, X76,65, X87,66, X67,94, X75,86, X88,77, X78,93]]
```

Out[ ]=

pdf

$$\left\{170.313,\ \left\{-\frac{\left(-1+2\,T-T^2-T^3+2\,T^4-T^5+T^8\right)\,\left(-1+T^3-2\,T^4+T^5+T^6-2\,T^7+T^8\right)}{T^8},\right.\right.$$
$$\frac{1}{T^{16}}\,(-1+T)^2\,\left(5-18\,T+33\,T^2-32\,T^3+2\,T^4+42\,T^5-62\,T^6-8\,T^7+166\,T^8-242\,T^9+108\,T^{10}+\right.$$
$$\left.\left.\left.132\,T^{11}-226\,T^{12}+148\,T^{13}-11\,T^{14}-36\,T^{15}-11\,T^{16}+148\,T^{17}-226\,T^{18}+132\,T^{19}+108\,T^{20}-\right.\right.\right.$$
$$\left.\left.\left.242\,T^{21}+166\,T^{22}-8\,T^{23}-62\,T^{24}+42\,T^{25}+2\,T^{26}-32\,T^{27}+33\,T^{28}-18\,T^{29}+5\,T^{30}\right)\right\}\right\}$$

pdf

Strong!

pdf

```
{NumberOfKnots[{3, 12}],
 Length@Union@Table[Z[K], {K, AllKnots[{3, 12}]}],
 Length@Union@Table[{HOMFLYPT[K], Kh[K]}, {K, AllKnots[{3, 12}]}]}
```

Out[ ]=

pdf

{2977, 2882, 2785}

In[ ]:=
```
2977 - {2882, 2785}
```

Out[ ]=

{95, 192}

tex

So the pair (Δ,ρ_1) attains 2,882 distinct values on the 2,977 prime knots with up to 12 crossings (a deficit of 95), whereas the pair (HOMFLYPT, Khovanov Homology) attains only 2,785 distinct values on the same knots (a deficit of 192).

tex

\def\nbpdfText#1{\vskip 1mm\par\noindent\includegraphics[width=\linewidth]{#1}}

pdf

tex

\def\nbpdfText#1{\vskip 1mm\par\noindent\includegraphics{#1}}

Invariance under R3

exec

```
nb2tex$TeXFileName = "Invariance.tex";
```

pdf

In[•]:=
```
δ_,j_ := If[i === j, 1, 0];
gRules_s_,i_,j_ := {g_iβ_ :> δ_iβ + T^s g_i⁺,β + (1 - T^s) g_j⁺,β, g_jβ_ :> δ_jβ + g_j⁺,β,
  g_α_,i :> T^-s (g_α,i⁺ - δ_α,i⁺), g_α_,j :> g_α,j⁺ - (1 - T^s) g_αi - δ_α,j⁺}
```

Proof of Reidemeister 3:

pdf

In[•]:=
```
lhs = R1[1, j, k] + R1[1, i, k⁺] + R1[1, i⁺, j⁺] //. gRules_1,j,k ⋃ gRules_1,i,k⁺ ⋃ gRules_1,i⁺,j⁺;
rhs = R1[1, i, j] + R1[1, i⁺, k] + R1[1, j⁺, k⁺] //. gRules_1,i,j ⋃ gRules_1,i⁺,k ⋃ gRules_1,j⁺,k⁺;
Simplify[lhs == rhs]
```

Out[•]=
pdf

```
True
```

tex

Next comes Reid1, where we use results from an earlier example:

In[•]:=
$$\begin{pmatrix} 1 & T^{-1} & 1 \\ 0 & T^{-1} & 1 \\ 0 & 0 & 1 \end{pmatrix} // \text{Inverse} // \text{MatrixForm}$$

Out[•]//MatrixForm=
$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & T & -T \\ 0 & 0 & 1 \end{pmatrix}$$

pdf

In[•]:=
$$R_1[1,2,1] - 1\,(g_{22} - 1/2) \;/.\; g_{\alpha\_,\beta\_} :\to \begin{pmatrix} 1 & T^{-1} & 1 \\ 0 & T^{-1} & 1 \\ 0 & 0 & 1 \end{pmatrix} [\![\alpha, \beta]\!]$$

Out[•]=
pdf

$$\frac{1}{T^2} - \frac{1}{T} - \frac{-1 + \frac{1}{T}}{T}$$

tex

Invariance under the other moves is proven similarly.

exec

```
nb2tex$TeXFileName = "Rhod.tex";
nb2tex$PDFWidth = 4.2 / 0.7;
```

On to ρ_d!

tex

{\bf\red Implementation.} Data, then program (with output using the \text{Conway} variable $z=\sqrt{T}-1/\sqrt{T}$), and then a demo. See {\tt Rho.nb} of \web{ap}.
\def\nbpdfInput#1{\vskip 1mm\par\noindent\includegraphics[scale=0.7]{#1}}
\def\nbpdfOutput#1{\vskip 1mm\par\noindent\includegraphics[scale=0.7]{#1}}

pdf

In[•]:=
```
V@γ1,φ_[k_] = φ (1 / 2 - p̄k x̄k); V@γ2,φ_[k_] = -φ^2 p̄k x̄k / 2; V@γ3,φ_[k_] := -φ^3 p̄k x̄k / 6
```

pdf

In[•]:=
```
V@r_{1,s_}[i_, j_] :=
 s (-1 + 2 p_i x_i - 2 p_j x_i + (-1 + T^s) p_i p_j x_i^2 + (1 - T^s) p_j^2 x_i^2 - 2 p_i p_j x_i x_j + 2 p_j^2 x_i x_j) / 2
```

pdf

In[•]:=
```
V@r_{2,1}[i_, j_] := (-6 p_i x_i + 6 p_j x_i - 3 (-1 + 3 T) p_i p_j x_i^2 + 3 (-1 + 3 T) p_j^2 x_i^2 + 4 (-1 + T) p_i^2 p_j x_i^3 -
    2 (-1 + T) (5 + T) p_i p_j^2 x_i^3 + 2 (-1 + T) (3 + T) p_j^3 x_i^3 + 18 p_i p_j x_i x_j - 18 p_j^2 x_i x_j -
    6 p_i^2 p_j x_i^2 x_j + 6 (2 + T) p_i p_j^2 x_i^2 x_j - 6 (1 + T) p_j^3 x_i^2 x_j - 6 p_i p_j^2 x_i x_j^2 + 6 p_j^3 x_i x_j^2) / 12
```

pdf

In[•]:=
```
V@r_{2,-1}[i_, j_] :=
 (-6 T^2 p_i x_i + 6 T^2 p_j x_i + 3 (-3 + T) T p_i p_j x_i^2 - 3 (-3 + T) T p_j^2 x_i^2 - 4 (-1 + T) T p_i^2 p_j x_i^3 + 2 (-1 + T)
     (1 + 5 T) p_i p_j^2 x_i^3 - 2 (-1 + T) (1 + 3 T) p_j^3 x_i^3 + 18 T^2 p_i p_j x_i x_j - 18 T^2 p_j^2 x_i x_j - 6 T^2 p_i^2 p_j x_i^2 x_j +
   6 T (1 + 2 T) p_i p_j^2 x_i^2 x_j - 6 T (1 + T) p_j^3 x_i^2 x_j - 6 T^2 p_i p_j^2 x_i x_j^2 + 6 T^2 p_j^3 x_i x_j^2) / (12 T^2)
```

pdf

In[•]:=
```
V@r_{3,1}[i_, j_] :=
 (4 p_i x_i - 4 p_j x_i + 2 (5 + 7 T) p_i p_j x_i^2 - 2 (5 + 7 T) p_j^2 x_i^2 - 4 (-5 + 6 T) p_i^2 p_j x_i^3 + 4 (-16 + 17 T + 2 T^2)
     p_i p_j^2 x_i^3 - 4 (-11 + 11 T + 2 T^2) p_j^3 x_i^3 + 3 (-1 + T) p_i^2 p_j x_i^4 - 3 (-1 + T) (4 + 3 T) p_i^2 p_j^2 x_i^4 +
   (-1 + T) (13 + 22 T + T^2) p_i p_j^3 x_i^4 - (-1 + T) (4 + 13 T + T^2) p_j^4 x_i^4 - 28 p_i p_j x_i x_j + 28 p_j^2 x_i x_j +
   36 p_i^2 p_j x_i^2 x_j - 12 (9 + 2 T) p_i p_j^2 x_i^2 x_j + 24 (3 + T) p_j^3 x_i^2 x_j - 4 p_i^3 p_j x_i^3 x_j + 28 T p_i^2 p_j^2 x_i^3 x_j -
   4 (-6 + 17 T + T^2) p_i p_j^3 x_i^3 x_j + 4 (-5 + 10 T + T^2) p_j^4 x_i^3 x_j + 24 p_i p_j^2 x_i x_j^2 - 24 p_j^3 x_i x_j^2 -
   24 p_i^2 p_j^2 x_i^2 x_j^2 + 6 (10 + T) p_i p_j^3 x_i^2 x_j^2 - 6 (6 + T) p_j^4 x_i^2 x_j^2 - 4 p_i p_j^3 x_i x_j^3 + 4 p_j^4 x_i x_j^3) / 24
```

pdf

In[•]:=
```
V@r_{3,-1}[i_, j_] :=
 (-4 T^3 p_i x_i + 4 T^3 p_j x_i - 2 T^2 (7 + 5 T) p_i p_j x_i^2 + 2 T^2 (7 + 5 T) p_j^2 x_i^2 - 4 T^2 (-6 + 5 T) p_i^2 p_j x_i^3 +
   4 T (-2 - 17 T + 16 T^2) p_i p_j^2 x_i^3 - 4 T (-2 - 11 T + 11 T^2) p_j^3 x_i^3 + 3 (-1 + T) T^2 p_i^3 p_j x_i^4 -
   3 (-1 + T) T (3 + 4 T) p_i^2 p_j^2 x_i^4 + (-1 + T) (1 + 22 T + 13 T^2) p_i p_j^3 x_i^4 -
   (-1 + T) (1 + 13 T + 4 T^2) p_j^4 x_i^4 + 28 T^3 p_i p_j x_i x_j - 28 T^3 p_j^2 x_i x_j - 36 T^3 p_i^2 p_j x_i^2 x_j +
   12 T^2 (2 + 9 T) p_i p_j^2 x_i^2 x_j - 24 T^2 (1 + 3 T) p_j^3 x_i^2 x_j + 4 T^3 p_i^3 p_j x_i^3 x_j -
   28 T^2 p_i^2 p_j^2 x_i^3 x_j - 4 T (-1 - 17 T + 6 T^2) p_i p_j^3 x_i^3 x_j + 4 T (-1 - 10 T + 5 T^2) p_j^4 x_i^3 x_j -
   24 T^3 p_i p_j^2 x_i x_j^2 + 24 T^3 p_j^3 x_i x_j^2 + 24 T^3 p_i^2 p_j^2 x_i^2 x_j^2 - 6 T^2 (1 + 10 T) p_i p_j^3 x_i^2 x_j^2 +
   6 T^2 (1 + 6 T) p_j^4 x_i^2 x_j^2 + 4 T^3 p_i p_j^3 x_i x_j^3 - 4 T^3 p_j^4 x_i x_j^3) / (24 T^3)
```

pdf

In[•]:=
```
{p^*, x^*, \[Overscript]p^*, \[Overscript]x^*} = {π, ξ, \[Overscript]π, \[Overscript]ξ};  (z_{i__})^* := (z^*)_i;
Zip_{}[ℰ_] := ℰ;
Zip_{z_,zs___}[ℰ_] := (Collect[ℰ // Zip_{zs}, z] /. f_. z^{d_.} :> (D[f, {z^*, d}])) /. z^* → 0
```

pdf

In[•]:=
```
gPair[fs_, w_] := gPair[fs, w] = Collect[Zip_{Join@@Table[{p_α, p̄_α, x_α, x̄_α}, {α, w}]}[(Times @@ (V /@ fs)) Exp[Sum[g_{α,β} (π_α + π̄_α) (ξ_β + ξ̄_β), {α, w}, {β, w}] - Sum[ξ̄_α π_α, {α, w}]]], g__, Factor]
```

pdf

In[•]:=
```
T2z[p_] := Module[{q = Expand[p], n, c},
  If[q === 0, 0, c = Coefficient[q, T, n = Exponent[q, T]];
   c z^(2n) + T2z[q - c (T^(1/2) - T^(-1/2))^(2n)]]];
```

pdf

In[•]:=
```
Z_d_[K_] := Module[{Cs, φ, n, A, s, i, j, k, Δ, G, d1, Z1, Z2, Z3},
  {Cs, φ} = Rot[K]; n = Length[Cs]; A = IdentityMatrix[2n + 1];
  Cases[Cs, {s_, i_, j_} :> (A[[{i, j}, {i + 1, j + 1}]] += ({{-T^s, T^s - 1}, {0, -1}}))];
  {Δ, G} = Factor@{T^((-Total[φ] - Total[Cs[[All, 1]]])/2) Det@A, Inverse@A};
  Z1 = Exp[Total[Cases[Cs, {s_, i_, j_} :> Sum[ϵ^d1 r_{d1,s}[i, j], {d1, d}]]] +
      Sum[ϵ^d1 γ_{d1,φ[[k]]}[k], {k, 2n}, {d1, d}] /. γ_,0[_] → 0];
  Z2 = Expand[F[{}, {}] × Normal@Series[Z1, {ϵ, 0, d}]] //. F[fs_, {es___}] ×
      (f : (r | γ)_ps__[is__])^p_. :> F[Join[fs, Table[f, p]], DeleteDuplicates@{es, is}];
  Z3 = Expand[Z2 /. F[fs_, es_] :> Expand[gPair[
        Replace[fs, Thread[es → Range@Length@es], {2}], Length@es
       ] /. g_α_,β_ :> G[[es[[α]], es[[β]]]]]];
  Collect[{Δ, Z3 /. ϵ^p_. → p! Δ^(2p) ϵ^p}, ϵ, T2z]];
```

In[•]:=
```
Z_3[Knot[3, 1]] // Timing
```

KnotTheory: Loading precomputed data in PD4Knots`.

Out[•]=
$$\left\{49.9844,\ \left\{1+z^2,\ 1+\left(2z^2+z^4\right)\epsilon+\left(2-4z^2+3z^4+4z^6+z^8\right)\epsilon^2+\left(-12+74z^2-27z^4-20z^6+8z^8+6z^{10}+z^{12}\right)\epsilon^3\right\}\right\}$$

In[•]:=
```
Z_3[Knot[3, 1]] // Timing
```

Out[•]=
$$\left\{1.26563,\ \left\{1+z^2,\ 1+\left(2z^2+z^4\right)\epsilon+\left(2-4z^2+3z^4+4z^6+z^8\right)\epsilon^2+\left(-12+74z^2-27z^4-20z^6+8z^8+6z^{10}+z^{12}\right)\epsilon^3\right\}\right\}$$

Demos

exec
```
nb2tex$PDFWidth = 8 / 0.75;
```

tex
```
\end{multicols}
\def\nbpdfInput#1{\vskip 1mm\par\noindent\includegraphics[scale=0.75]{#1}}
```

\def\nbpdfOutput#1{\vskip 1mm\par\noindent\includegraphics[scale=0.75]{#1}}

In[•]:=
```
GST48 = EPD[X14,1, X2,29, X3,40, X43,4, X26,5, X6,95, X96,7, X13,8, X9,28, X10,41, X42,11, X27,12,
   X30,15, X16,61, X17,72, X18,83, X19,34, X89,20, X21,92, X79,22, X68,23, X57,24, X25,56, X62,31,
   X73,32, X84,33, X50,35, X36,81, X37,70, X38,59, X39,54, X44,55, X58,45, X69,46, X80,47, X48,91,
   X90,49, X51,82, X52,71, X53,60, X63,74, X64,85, X76,65, X87,66, X67,94, X75,86, X88,77, X78,93];
Z2[GST48] // Timing
Z2[GST48] // Timing
```

Out[•]=

$$\{564.578,\ \{1-4z^2-61z^4-207z^6-296z^8-210z^{10}-77z^{12}-14z^{14}-z^{16},$$
$$1+\left(38z^2+255z^4+1696z^6+16\,281z^8+86\,952z^{10}+259\,994z^{12}+487\,372z^{14}+615\,066z^{16}+\right.$$
$$\left.543\,148z^{18}+341\,714z^{20}+153\,722z^{22}+48\,983z^{24}+10\,776z^{26}+1554z^{28}+132z^{30}+5z^{32}\right)\epsilon+$$
$$\left(-8-484z^2+9709z^4+165\,952z^6+1\,590\,491z^8+16\,256\,508z^{10}+115\,341\,797z^{12}+432\,685\,748z^{14}+\right.$$
$$395\,838\,354z^{16}-4\,017\,557\,792z^{18}-23\,300\,064\,167z^{20}-70\,082\,264\,972z^{22}-142\,572\,271\,191z^{24}-$$
$$209\,475\,503\,700z^{26}-221\,616\,295\,209z^{28}-151\,502\,648\,428z^{30}-23\,700\,199\,243z^{32}+$$
$$99\,462\,146\,328z^{34}+164\,920\,463\,074z^{36}+162\,550\,825\,432z^{38}+119\,164\,552\,296z^{40}+$$
$$69\,153\,062\,608z^{42}+32\,547\,596\,611z^{44}+12\,541\,195\,448z^{46}+3\,961\,384\,155z^{48}+1\,021\,219\,696z^{50}+$$
$$\left.212\,773\,106z^{52}+35\,264\,208z^{54}+4\,537\,548z^{56}+436\,600z^{58}+29\,536z^{60}+1252z^{62}+25z^{64}\right)\epsilon^2\}\}$$

Out[•]=

$$\{598.109,\ \{1-4z^2-61z^4-207z^6-296z^8-210z^{10}-77z^{12}-14z^{14}-z^{16},$$
$$1+\left(38z^2+255z^4+1696z^6+16\,281z^8+86\,952z^{10}+259\,994z^{12}+487\,372z^{14}+615\,066z^{16}+\right.$$
$$\left.543\,148z^{18}+341\,714z^{20}+153\,722z^{22}+48\,983z^{24}+10\,776z^{26}+1554z^{28}+132z^{30}+5z^{32}\right)\epsilon+$$
$$\left(-8-484z^2+9709z^4+165\,952z^6+1\,590\,491z^8+16\,256\,508z^{10}+115\,341\,797z^{12}+432\,685\,748z^{14}+\right.$$
$$395\,838\,354z^{16}-4\,017\,557\,792z^{18}-23\,300\,064\,167z^{20}-70\,082\,264\,972z^{22}-142\,572\,271\,191z^{24}-$$
$$209\,475\,503\,700z^{26}-221\,616\,295\,209z^{28}-151\,502\,648\,428z^{30}-23\,700\,199\,243z^{32}+$$
$$99\,462\,146\,328z^{34}+164\,920\,463\,074z^{36}+162\,550\,825\,432z^{38}+119\,164\,552\,296z^{40}+$$
$$69\,153\,062\,608z^{42}+32\,547\,596\,611z^{44}+12\,541\,195\,448z^{46}+3\,961\,384\,155z^{48}+1\,021\,219\,696z^{50}+$$
$$\left.212\,773\,106z^{52}+35\,264\,208z^{54}+4\,537\,548z^{56}+436\,600z^{58}+29\,536z^{60}+1252z^{62}+25z^{64}\right)\epsilon^2\}\}$$

pdf
```
Z2[GST48] (* takes a few minutes *)
```

Out[•]=
pdf

$$\{1-4z^2-61z^4-207z^6-296z^8-210z^{10}-77z^{12}-14z^{14}-z^{16},$$
$$1+\left(38z^2+255z^4+1696z^6+16\,281z^8+86\,952z^{10}+259\,994z^{12}+487\,372z^{14}+615\,066z^{16}+\right.$$
$$\left.543\,148z^{18}+341\,714z^{20}+153\,722z^{22}+48\,983z^{24}+10\,776z^{26}+1554z^{28}+132z^{30}+5z^{32}\right)\epsilon+$$
$$\left(-8-484z^2+9709z^4+165\,952z^6+1\,590\,491z^8+16\,256\,508z^{10}+115\,341\,797z^{12}+432\,685\,748z^{14}+\right.$$
$$395\,838\,354z^{16}-4\,017\,557\,792z^{18}-23\,300\,064\,167z^{20}-70\,082\,264\,972z^{22}-142\,572\,271\,191z^{24}-$$
$$209\,475\,503\,700z^{26}-221\,616\,295\,209z^{28}-151\,502\,648\,428z^{30}-23\,700\,199\,243z^{32}+$$
$$99\,462\,146\,328z^{34}+164\,920\,463\,074z^{36}+162\,550\,825\,432z^{38}+119\,164\,552\,296z^{40}+$$
$$69\,153\,062\,608z^{42}+32\,547\,596\,611z^{44}+12\,541\,195\,448z^{46}+3\,961\,384\,155z^{48}+1\,021\,219\,696z^{50}+$$
$$\left.212\,773\,106z^{52}+35\,264\,208z^{54}+4\,537\,548z^{56}+436\,600z^{58}+29\,536z^{60}+1252z^{62}+25z^{64}\right)\epsilon^2\}$$

In[•]:=
```
Table[Join[{K[[1]]_K[[2]]}, Z_3[K]], {K, AllKnots[{3, 6}]}] // Timing
```

Out[•]=

$$\{256.063, \{\{3_1, 1+z^2, 1+\left(2z^2+z^4\right)\in+\left(2-4z^2+3z^4+4z^6+z^8\right)\in^2+\left(-12+74z^2-27z^4-20z^6+8z^8+6z^{10}+z^{12}\right)\in^3\},$$
$$\{4_1, 1-z^2, 1+\left(-2+2z^4\right)\in^2\}, \{5_1, 1+3z^2+z^4, 1+\left(10z^2+21z^4+12z^6+2z^8\right)\in+$$
$$\left(6-28z^2+33z^4+364z^6+655z^8+536z^{10}+227z^{12}+48z^{14}+4z^{16}\right)\in^2+$$
$$\left(-60+970z^2+645z^4-3380z^6-3280z^8+7470z^{10}+19\,475z^{12}+\right.$$
$$\left.20\,536z^{14}+12\,564z^{16}+4774z^{18}+1109z^{20}+144z^{22}+8z^{24}\right)\in^3\},$$
$$\{5_2, 1+2z^2, 1+\left(6z^2+5z^4\right)\in+\left(4-20z^2+43z^4+64z^6+26z^8\right)\in^2+$$
$$\left(-36+498z^2-883z^4+100z^6+816z^8+556z^{10}+146z^{12}\right)\in^3\},$$
$$\{6_1, 1-2z^2, 1+\left(-2z^2+z^4\right)\in+\left(-4+4z^2+25z^4-8z^6+2z^8\right)\in^2+$$
$$\left(12+154z^2-223z^4-608z^6+100z^8-52z^{10}+10z^{12}\right)\in^3\},$$
$$\{6_2, 1-z^2-z^4, 1+\left(-2z^2-3z^4+2z^6+z^8\right)\in+$$
$$\left(-2-4z^2+29z^4+28z^6+42z^8-8z^{10}-2z^{12}+4z^{14}+z^{16}\right)\in^2+\left(12+166z^2+155z^4-\right.$$
$$\left.194z^6-2453z^8-1622z^{10}-1967z^{12}-258z^{14}+49z^{16}-30z^{18}+z^{20}+6z^{22}+z^{24}\right)\in^3\},$$
$$\{6_3, 1+z^2+z^4, 1+\left(2+8z^2-16z^6-24z^8-16z^{10}-2z^{12}\right)\in^2\}\}\}$$

In[•]:=
```
Table[Join[{K[[1]]_K[[2]]}, Z_3[K]], {K, AllKnots[{3, 6}]}] // Timing
```

Out[•]=

$$\{143.641, \{\{3_1, 1+z^2, 1+\left(2z^2+z^4\right)\in+\left(2-4z^2+3z^4+4z^6+z^8\right)\in^2+\left(-12+74z^2-27z^4-20z^6+8z^8+6z^{10}+z^{12}\right)\in^3\},$$
$$\{4_1, 1-z^2, 1+\left(-2+2z^4\right)\in^2\}, \{5_1, 1+3z^2+z^4, 1+\left(10z^2+21z^4+12z^6+2z^8\right)\in+$$
$$\left(6-28z^2+33z^4+364z^6+655z^8+536z^{10}+227z^{12}+48z^{14}+4z^{16}\right)\in^2+$$
$$\left(-60+970z^2+645z^4-3380z^6-3280z^8+7470z^{10}+19\,475z^{12}+\right.$$
$$\left.20\,536z^{14}+12\,564z^{16}+4774z^{18}+1109z^{20}+144z^{22}+8z^{24}\right)\in^3\},$$
$$\{5_2, 1+2z^2, 1+\left(6z^2+5z^4\right)\in+\left(4-20z^2+43z^4+64z^6+26z^8\right)\in^2+$$
$$\left(-36+498z^2-883z^4+100z^6+816z^8+556z^{10}+146z^{12}\right)\in^3\},$$
$$\{6_1, 1-2z^2, 1+\left(-2z^2+z^4\right)\in+\left(-4+4z^2+25z^4-8z^6+2z^8\right)\in^2+$$
$$\left(12+154z^2-223z^4-608z^6+100z^8-52z^{10}+10z^{12}\right)\in^3\},$$
$$\{6_2, 1-z^2-z^4, 1+\left(-2z^2-3z^4+2z^6+z^8\right)\in+$$
$$\left(-2-4z^2+29z^4+28z^6+42z^8-8z^{10}-2z^{12}+4z^{14}+z^{16}\right)\in^2+\left(12+166z^2+155z^4-\right.$$
$$\left.194z^6-2453z^8-1622z^{10}-1967z^{12}-258z^{14}+49z^{16}-30z^{18}+z^{20}+6z^{22}+z^{24}\right)\in^3\},$$
$$\{6_3, 1+z^2+z^4, 1+\left(2+8z^2-16z^6-24z^8-16z^{10}-2z^{12}\right)\in^2\}\}\}$$

tex

```
\def\nbpdfOutput#1{\vskip 1mm\par\noindent\includegraphics[width=\linewidth]{#1}}
```

pdf

```
TableForm[Table[Join[{K⟦1⟧_K⟦2⟧}, Z_3[K]], {K, AllKnots[{3, 6}]}], TableAlignments → Center]
(* takes a few minutes *)
```

pdf

KnotTheory: Loading precomputed data in PD4Knots`.

Out[ ]//TableForm=
pdf

3_1	$1+z^2$	$1+\left(2z^2+z^4\right)\epsilon$
4_1	$1-z^2$	
5_1	$1+3z^2+z^4$	$1+\left(10z^2+21z^4+12z^6+2z^8\right)\epsilon+\left(6-28z^2+33z^4+364z^6+655z^8+536z^{10}+227$
5_2	$1+2z^2$	$1+\left(6z^2+5z^4\right)\epsilon+\left(4-2$
6_1	$1-2z^2$	$1+\left(-2z^2+z^4\right)\epsilon+\left(-$
6_2	$1-z^2-z^4$	$1+\left(-2z^2-3z^4+2z^6+z^8\right)\epsilon+\left(-2-4z^2+29z^4+28z^6+42z^8-8$
6_3	$1+z^2+z^4$	