Pensieve header: A development notebook for ρ_k.

Preliminaries

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\Oaxaca-2210"];
Once[<< KnotTheory`; << Rot.m];
```

The Old Program

In[◦]:=
```
R1[s_, i_, j_] := s (g_{j i} (g_{j+, j} + g_{j, j+} - g_{i j}) - g_{i i} (g_{j, j+} - 1) - 1 / 2);
ρ[K_] := ρ[K] = Module[{Cs, φ, n, A, s, i, j, k, Δ, G, ρ1},
    {Cs, φ} = Rot[K]; n = Length[Cs];
    A = IdentityMatrix[2 n + 1];
    Cases[Cs, {s_, i_, j_} :> (A⟦{i, j}, {i + 1, j + 1}⟧ += ( -T^s  T^s - 1
                                                              0     -1    ))];
    Δ = T^((-Total[φ]-Total[Cs⟦All,1⟧])/2) Det[A];
    G = Inverse[A];
    ρ1 = Sum_{k=1}^{n} R1 @@ Cs⟦k⟧ - Sum_{k=1}^{2 n} φ⟦k⟧ (g_{kk} - 1 / 2);
    Factor@{Δ, Δ^2 ρ1 /. α_^+ :> α + 1 /. g_{α_, β_} :> G⟦α, β⟧}];
```

The g-Rules

In[◦]:=
```
δ_{i_, j_} := If[i === j, 1, 0];
gRules_{s_, i_, j_} := {g_{i β_} :> δ_{i β} + T^s g_{i+, β} + (1 - T^s) g_{j+, β}, g_{j β_} :> δ_{j β} + g_{j+, β},
   g_{α_, i} :> T^-s (g_{α, i+} - δ_{α, i+}), g_{α_ j} :> g_{α, j+} - (1 - T^s) g_{α i} - δ_{α, j+}}
(α_^+)^+ := α^"++";  (* this is for cosmetic reasons only *)
```

CF

In[◦]:=
```
CF[ℰ_] := Module[{vs = Union@Cases[ℰ, (g | p | x)__, ∞]}, Total[
    CoefficientRules[Expand@ℰ, vs] /. (ps_ → c_) :> Factor[c] (Times @@ vs^ps)
  ]]
```

g2px and px2g

In[◦]:=
```
g2px[ℰ_] := Module[{λ}, Expand[ℰ /. g_{α_, β_} :> λ p_α x_β] /. λ^k_. :> 1 / k!]
```

In[◦]:=
```
{p^*, x^*, π^*, ξ^*} = {π, ξ, p, x};  (u_{i_})^* := (u^*)_i;
```

In[∘]:=
```
Zip{}[ℰ_] := ℰ;
Zip{ζ_,ζs___}[ℰ_] := (Collect[ℰ // Zip{ζs}, ζ] /. f_. ζ^d_. :> (D[f, {ζ*, d}])) /. ζ* → 0
```

In[∘]:=
```
px2g[ℰ_] := Module[{ps, xs, Q},
  ps = Union[Cases[ℰ, p_α_, ∞]];
  xs = Union[Cases[ℰ, x_α_, ∞]];
  Q = Sum[p0* x0* g_{p0[[2]],x0[[2]]}, {p0, ps}, {x0, xs}];
  Zip_{ps⋃xs}[ℰ ⅇ^Q] // Expand
 ]
```

In[∘]:= `R_1[1, i, j] // Expand`

Out[∘]=

$$-\frac{1}{2}+g_{i,i}-g_{i,j}\,g_{j,i}-g_{i,i}\,g_{j,1+j}+g_{j,i}\,g_{j,1+j}+g_{j,i}\,g_{1+j,j}$$

In[∘]:= `R_1[1, i, j] // g2px`

Out[∘]=

$$-\frac{1}{2}+p_i\,x_i-\frac{1}{2}\,p_i\,p_j\,x_i\,x_j+\frac{1}{2}\,p_j\,p_{1+j}\,x_i\,x_j-\frac{1}{2}\,p_i\,p_j\,x_i\,x_{1+j}+\frac{1}{2}\,p_j^2\,x_i\,x_{1+j}$$

In[∘]:= `R_1[1, i, j] // g2px // px2g`

Out[∘]=

$$-\frac{1}{2}+g_{i,i}-\frac{1}{2}\,g_{i,j}\,g_{j,i}-\frac{1}{2}\,g_{i,1+j}\,g_{j,i}-\frac{1}{2}\,g_{i,i}\,g_{j,j}-\frac{1}{2}\,g_{i,i}\,g_{j,1+j}+g_{j,i}\,g_{j,1+j}+\frac{1}{2}\,g_{j,j}\,g_{1+j,i}+\frac{1}{2}\,g_{j,i}\,g_{1+j,j}$$

In[∘]:= `R_1[1, i, j] // g2px // px2g // g2px`

Out[∘]=

$$-\frac{1}{2}+p_i\,x_i-\frac{1}{2}\,p_i\,p_j\,x_i\,x_j+\frac{1}{2}\,p_j\,p_{1+j}\,x_i\,x_j-\frac{1}{2}\,p_i\,p_j\,x_i\,x_{1+j}+\frac{1}{2}\,p_j^2\,x_i\,x_{1+j}$$

In[∘]:= `(1 + (p_1 + p_2) (x_1 + x_2))^3 // px2g`

Out[∘]=

$$1+3\,g_{1,1}+6\,g_{1,1}^2+6\,g_{1,1}^3+3\,g_{1,2}+12\,g_{1,1}\,g_{1,2}+18\,g_{1,1}^2\,g_{1,2}+6\,g_{1,2}^2+18\,g_{1,1}\,g_{1,2}^2+6\,g_{1,2}^3+3\,g_{2,1}+$$
$$12\,g_{1,1}\,g_{2,1}+18\,g_{1,1}^2\,g_{2,1}+12\,g_{1,2}\,g_{2,1}+36\,g_{1,1}\,g_{1,2}\,g_{2,1}+18\,g_{1,2}^2\,g_{2,1}+6\,g_{2,1}^2+18\,g_{1,1}\,g_{2,1}^2+18\,g_{1,2}\,g_{2,1}^2+$$
$$6\,g_{2,1}^3+3\,g_{2,2}+12\,g_{1,1}\,g_{2,2}+18\,g_{1,1}^2\,g_{2,2}+12\,g_{1,2}\,g_{2,2}+36\,g_{1,1}\,g_{1,2}\,g_{2,2}+18\,g_{1,2}^2\,g_{2,2}+12\,g_{2,1}\,g_{2,2}+$$
$$36\,g_{1,1}\,g_{2,1}\,g_{2,2}+36\,g_{1,2}\,g_{2,1}\,g_{2,2}+18\,g_{2,1}^2\,g_{2,2}+6\,g_{2,2}^2+18\,g_{1,1}\,g_{2,2}^2+18\,g_{1,2}\,g_{2,2}^2+18\,g_{2,1}\,g_{2,2}^2+6\,g_{2,2}^3$$

In[∘]:= `(1 + (p_1 + p_2) (x_1 + x_2))^4 // px2g // g2px // FullSimplify`

Out[∘]=

$$(1+(p_1+p_2)\,(x_1+x_2))^4$$

`R_1[1, i, j] R_1[1, k, l] // g2px // px2g // FullSimplify // Short`

$$\frac{1}{24}\Big(-4\,g_{i,k}\,g_{j,1+j}\,g_{k,i}-4\,g_{i,j}\,g_{j,k}\,g_{k,i}-$$
$$4\,g_{i,1+j}\,g_{j,k}\,g_{k,i}+\ll 276\gg+(\ll 1\gg)\,g_{1+l,l}+g_{i,i}\,(\ll 1\gg)+g_{j,i}\,(\ll 1\gg)\Big)$$

```
(R1[1, 1, 3] R1[1, 5, 7] //. gRules1,1,3 ∪ gRules1,5,7) // g2px // px2g // FullSimplify // Short
```

$$\frac{6\,T^4 - 6\,T^3\, g_{\ll 1 \gg}\, g_{\ll 1 \gg} + \ll 991 \gg + g_{3^+,1^+}\left(12\,(-1+T)\,T^3 + \ll 639 \gg\right)}{24\,T^4}$$

Generic Perturbations

In[∘]:=

```
Module[{i, j, k},
 AllMonomials[{}, 0] = {1};
 AllMonomials[{}, d_Integer] /; d > 0 := {};
 AllMonomials[{v_, vs___}, d_Integer] :=
  Join @@ Table[v^(d-k) AllMonomials[{vs}, k], {k, 0, d}];
 AllMonomials[vs_List, {d_}] := Join @@ Table[AllMonomials[vs, k], {k, 0, d}];
 Basis[js_List, m_] := Flatten@Outer[Times,
    AllMonomials[Table[p_j, {j, js}], m], AllMonomials[Table[x_j, {j, js}], m]];
 Basis[js_List, {m_}] := Flatten@Table[Basis[js, k], {k, 0, m}];
 GenericCombination[bas_, c_] := bas.Table[c_j, {j, Length@bas}];
 GenericCombination[bas_, c_k_] := bas.Table[c_{k,j}, {j, Length@bas}];
]
```

In[∘]:=

```
Module[{k},
  r_d_ [i_, j_] :=
   Expand@Together@Sum[ϵ^k GenericCombination[Basis[{i, j}, {k + 1}], c_k], {k, d}]
 ];
```

In[∘]:=

```
r2[j, k]
```

Out[∘]=

$$\begin{aligned}
&\epsilon\,\big(c_{1,1} + p_j x_j c_{1,2} + p_j x_k c_{1,3} + p_k x_j c_{1,4} + p_k x_k c_{1,5} + p_j^2 x_j^2 c_{1,6} + p_j^2 x_j x_k c_{1,7} + p_j^2 x_k^2 c_{1,8} + \\
&\quad p_j p_k x_j^2 c_{1,9} + p_j p_k x_j x_k c_{1,10} + p_j p_k x_k^2 c_{1,11} + p_k^2 x_j^2 c_{1,12} + p_k^2 x_j x_k c_{1,13} + p_k^2 x_k^2 c_{1,14}\big) + \\
&\epsilon^2\,\big(c_{2,1} + p_j x_j c_{2,2} + p_j x_k c_{2,3} + p_k x_j c_{2,4} + p_k x_k c_{2,5} + p_j^2 x_j^2 c_{2,6} + p_j^2 x_j x_k c_{2,7} + \\
&\quad p_j^2 x_k^2 c_{2,8} + p_j p_k x_j^2 c_{2,9} + p_j p_k x_j x_k c_{2,10} + p_j p_k x_k^2 c_{2,11} + p_k^2 x_j^2 c_{2,12} + p_k^2 x_j x_k c_{2,13} + \\
&\quad p_k^2 x_k^2 c_{2,14} + p_j^3 x_j^3 c_{2,15} + p_j^3 x_j^2 x_k c_{2,16} + p_j^3 x_j x_k^2 c_{2,17} + p_j^3 x_k^3 c_{2,18} + p_j^2 p_k x_j^3 c_{2,19} + \\
&\quad p_j^2 p_k x_j^2 x_k c_{2,20} + p_j^2 p_k x_j x_k^2 c_{2,21} + p_j^2 p_k x_k^3 c_{2,22} + p_j p_k^2 x_j^3 c_{2,23} + p_j p_k^2 x_j^2 x_k c_{2,24} + \\
&\quad p_j p_k^2 x_j x_k^2 c_{2,25} + p_j p_k^2 x_k^3 c_{2,26} + p_k^3 x_j^3 c_{2,27} + p_k^3 x_j^2 x_k c_{2,28} + p_k^3 x_j x_k^2 c_{2,29} + p_k^3 x_k^3 c_{2,30}\big)
\end{aligned}$$

In[ ]:=

```
Module[{d = 2, es = {i, j, k, i⁺, j⁺, k⁺}},
 Times[
   Normal@Series[Exp[r_d[j, k] + r_d[i, k⁺] + r_d[i⁺, j⁺]], {ϵ, 0, d}],
   Exp[Sum[g_{α,β} π_α ξ_β, {α, es}, {β, es}]]
  ] // Zip_{(p_#&/@es)⋃(x_#&/@es)} // Expand
]
```

Out[ ]=

$1 + 3 \in c_{1,1} + \frac{9}{2} \in^2 c_{1,1}^2 + 3 \in^2 c_{2,1} + \in c_{1,2}\, g_{i,i} + 3 \in^2 c_{1,1}\, c_{1,2}\, g_{i,i} + \in^2 c_{2,2}\, g_{i,i} + \cdots 2681 \cdots + 6 \in^2 c_{1,5}\, c_{1,14}\, g_{k^+,k^+}^3 + 6 \in^2 c_{2,30}\, g_{k^+,k^+}^3 + 6 \in^2 c_{1,11}\, c_{1,13}\, g_{i,i}\, g_{k^+,k^+}^3 + 6 \in^2 c_{1,10}\, c_{1,14}\, g_{i,i}\, g_{k^+,k^+}^3 + 24 \in^2 c_{1,11}\, c_{1,14}\, g_{i,k^+}\, g_{k^+,k^+}^3 + 24 \in^2 c_{1,13}\, c_{1,14}\, g_{k^+,i}\, g_{k^+,k^+}^3 + 12 \in^2 c_{1,14}^2\, g_{k^+,k^+}^4$

Full expression not available (original memory size: 2 MB)

In[ ]:=

```
Module[{d = 2, es = {i, j, k, i⁺, j⁺, k⁺}},
  Times[
    Normal@Series[Exp[r_d[j, k] + r_d[i, k⁺] + r_d[i⁺, j⁺]], {ϵ, 0, d}],
    Exp[Sum[g_{α,β} π_α ξ_β, {α, es}, {β, es}]]
   ] // Zip_{(p_#&/@es)⋃(x_#&/@es)} // Expand
 ] /. c_{1,_} → 0
```

Out[ ]=

$$1 + 3 \in^2 c_{2,1} + \in^2 c_{2,2} g_{i,i} + 2 \in^2 c_{2,6} g_{i,i}^2 + 6 \in^2 c_{2,15} g_{i,i}^3 + \in^2 c_{2,3} g_{i,k^+} + 2 \in^2 c_{2,7} g_{i,i} g_{i,k^+} +$$
$$6 \in^2 c_{2,16} g_{i,i}^2 g_{i,k^+} + 2 \in^2 c_{2,8} g_{i,k^+}^2 + 6 \in^2 c_{2,17} g_{i,i} g_{i,k^+}^2 + 6 \in^2 c_{2,18} g_{i,k^+}^3 + \in^2 c_{2,2} g_{j,j} +$$
$$2 \in^2 c_{2,6} g_{j,j}^2 + 6 \in^2 c_{2,15} g_{j,j}^3 + \in^2 c_{2,3} g_{j,k} + 2 \in^2 c_{2,7} g_{j,j} g_{j,k} + 6 \in^2 c_{2,16} g_{j,j}^2 g_{j,k} + 2 \in^2 c_{2,8} g_{j,k}^2 +$$
$$6 \in^2 c_{2,17} g_{j,j} g_{j,k}^2 + 6 \in^2 c_{2,18} g_{j,k}^3 + \in^2 c_{2,4} g_{k,j} + 2 \in^2 c_{2,9} g_{j,j} g_{k,j} + 6 \in^2 c_{2,19} g_{j,j}^2 g_{k,j} +$$
$$\in^2 c_{2,10} g_{j,k} g_{k,j} + 4 \in^2 c_{2,20} g_{j,j} g_{j,k} g_{k,j} + 2 \in^2 c_{2,21} g_{j,k}^2 g_{k,j} + 2 \in^2 c_{2,12} g_{k,j}^2 + 6 \in^2 c_{2,23} g_{j,j} g_{k,j}^2 +$$
$$2 \in^2 c_{2,24} g_{j,k} g_{k,j}^2 + 6 \in^2 c_{2,27} g_{k,j}^3 + \in^2 c_{2,5} g_{k,k} + \in^2 c_{2,10} g_{j,j} g_{k,k} + 2 \in^2 c_{2,20} g_{j,j}^2 g_{k,k} +$$
$$2 \in^2 c_{2,11} g_{j,k} g_{k,k} + 4 \in^2 c_{2,21} g_{j,j} g_{j,k} g_{k,k} + 6 \in^2 c_{2,22} g_{j,k}^2 g_{k,k} + 2 \in^2 c_{2,13} g_{k,j} g_{k,k} +$$
$$4 \in^2 c_{2,24} g_{j,j} g_{k,j} g_{k,k} + 4 \in^2 c_{2,25} g_{j,k} g_{k,j} g_{k,k} + 6 \in^2 c_{2,28} g_{k,j}^2 g_{k,k} + 2 \in^2 c_{2,14} g_{k,k}^2 +$$
$$2 \in^2 c_{2,25} g_{j,j} g_{k,k}^2 + 6 \in^2 c_{2,26} g_{j,k} g_{k,k}^2 + 6 \in^2 c_{2,29} g_{k,j} g_{k,k}^2 + 6 \in^2 c_{2,30} g_{k,k}^3 + \in^2 c_{2,2} g_{i^+,i^+} +$$
$$2 \in^2 c_{2,6} g_{i^+,i^+}^2 + 6 \in^2 c_{2,15} g_{i^+,i^+}^3 + \in^2 c_{2,3} g_{i^+,j^+} + 2 \in^2 c_{2,7} g_{i^+,i^+} g_{i^+,j^+} + 6 \in^2 c_{2,16} g_{i^+,i^+}^2 g_{i^+,j^+} +$$
$$2 \in^2 c_{2,8} g_{i^+,j^+}^2 + 6 \in^2 c_{2,17} g_{i^+,i^+} g_{i^+,j^+}^2 + 6 \in^2 c_{2,18} g_{i^+,j^+}^3 + \in^2 c_{2,4} g_{j^+,i^+} + 2 \in^2 c_{2,9} g_{i^+,i^+} g_{j^+,i^+} +$$
$$6 \in^2 c_{2,19} g_{i^+,i^+}^2 g_{j^+,i^+} + \in^2 c_{2,10} g_{i^+,j^+} g_{j^+,i^+} + 4 \in^2 c_{2,20} g_{i^+,i^+} g_{i^+,j^+} g_{j^+,i^+} + 2 \in^2 c_{2,21} g_{i^+,j^+}^2 g_{j^+,i^+} +$$
$$2 \in^2 c_{2,12} g_{j^+,i^+}^2 + 6 \in^2 c_{2,23} g_{i^+,i^+} g_{j^+,i^+}^2 + 2 \in^2 c_{2,24} g_{i^+,j^+} g_{j^+,i^+}^2 + 6 \in^2 c_{2,27} g_{j^+,i^+}^3 + \in^2 c_{2,5} g_{j^+,j^+} +$$
$$\in^2 c_{2,10} g_{i^+,i^+} g_{j^+,j^+} + 2 \in^2 c_{2,20} g_{i^+,i^+}^2 g_{j^+,j^+} + 2 \in^2 c_{2,11} g_{i^+,j^+} g_{j^+,j^+} + 4 \in^2 c_{2,21} g_{i^+,i^+} g_{i^+,j^+} g_{j^+,j^+} +$$
$$6 \in^2 c_{2,22} g_{i^+,j^+}^2 g_{j^+,j^+} + 2 \in^2 c_{2,13} g_{j^+,i^+} g_{j^+,j^+} + 4 \in^2 c_{2,24} g_{i^+,i^+} g_{j^+,i^+} g_{j^+,j^+} + 4 \in^2 c_{2,25} g_{i^+,j^+} g_{j^+,i^+} g_{j^+,j^+} +$$
$$6 \in^2 c_{2,28} g_{j^+,i^+}^2 g_{j^+,j^+} + 2 \in^2 c_{2,14} g_{j^+,j^+}^2 + 2 \in^2 c_{2,25} g_{i^+,i^+} g_{j^+,j^+}^2 + 6 \in^2 c_{2,26} g_{i^+,j^+} g_{j^+,j^+}^2 +$$
$$6 \in^2 c_{2,29} g_{j^+,i^+} g_{j^+,j^+}^2 + 6 \in^2 c_{2,30} g_{j^+,j^+}^3 + \in^2 c_{2,4} g_{k^+,i} + 2 \in^2 c_{2,9} g_{i,i} g_{k^+,i} + 6 \in^2 c_{2,19} g_{i,i}^2 g_{k^+,i} +$$
$$\in^2 c_{2,10} g_{i,k^+} g_{k^+,i} + 4 \in^2 c_{2,20} g_{i,i} g_{i,k^+} g_{k^+,i} + 2 \in^2 c_{2,21} g_{i,k^+}^2 g_{k^+,i} + 2 \in^2 c_{2,12} g_{k^+,i}^2 +$$
$$6 \in^2 c_{2,23} g_{i,i} g_{k^+,i}^2 + 2 \in^2 c_{2,24} g_{i,k^+} g_{k^+,i}^2 + 6 \in^2 c_{2,27} g_{k^+,i}^3 + \in^2 c_{2,5} g_{k^+,k^+} + \in^2 c_{2,10} g_{i,i} g_{k^+,k^+} +$$
$$2 \in^2 c_{2,20} g_{i,i}^2 g_{k^+,k^+} + 2 \in^2 c_{2,11} g_{i,k^+} g_{k^+,k^+} + 4 \in^2 c_{2,21} g_{i,i} g_{i,k^+} g_{k^+,k^+} + 6 \in^2 c_{2,22} g_{i,k^+}^2 g_{k^+,k^+} +$$
$$2 \in^2 c_{2,13} g_{k^+,i} g_{k^+,k^+} + 4 \in^2 c_{2,24} g_{i,i} g_{k^+,i} g_{k^+,k^+} + 4 \in^2 c_{2,25} g_{i,k^+} g_{k^+,i} g_{k^+,k^+} + 6 \in^2 c_{2,28} g_{k^+,i}^2 g_{k^+,k^+} +$$
$$2 \in^2 c_{2,14} g_{k^+,k^+}^2 + 2 \in^2 c_{2,25} g_{i,i} g_{k^+,k^+}^2 + 6 \in^2 c_{2,26} g_{i,k^+} g_{k^+,k^+}^2 + 6 \in^2 c_{2,29} g_{k^+,i} g_{k^+,k^+}^2 + 6 \in^2 c_{2,30} g_{k^+,k^+}^3$$

In[∘]:=
```
Module[{d = 2, es = {i, j, k, i⁺, j⁺, k⁺}, λ, λs},
 λs = λ_# & /@ Range[d - 1];
 Times[
   Normal@Series[Exp[r_d[j, k] + r_d[i, k⁺] + r_d[i⁺, j⁺]], {ϵ, 0, d}],
   Exp[Sum[g_{α,β} π_α ξ_β, {α, es}, {β, es}] + Sum[g_{α,β} π_α ξ_β + g_{β,α} π_β ξ_α, {α, es}, {β, λs}]]
  ] // Zip_{(p_#&/@es)⋃(x_#&/@es)} // Expand
]
```

Out[∘]=

$1+3\epsilon c_{1,1}+\frac{9}{2}\epsilon^2 c_{1,1}^2+3\epsilon^2 c_{2,1}+\epsilon c_{1,2} g_{i,i}+$ … 21 121 … $+$
$24\epsilon^2 c_{1,13} c_{1,14} g_{k^+,i} g^3_{k^+,k^+}+24\epsilon^2 \pi_{\lambda\$58174_1} \xi_{\lambda\$58174_1} c_{1,13} c_{1,14} g_{\lambda\$58174_1,i} g_{k^+,\lambda\$58174_1} g^3_{k^+,k^+}+$
$48\epsilon^2 \pi_{\lambda\$58174_1} \xi_{\lambda\$58174_1} c^2_{1,14} g_{\lambda\$58174_1,k^+} g_{k^+,\lambda\$58174_1} g^3_{k^+,k^+}+12\epsilon^2 c^2_{1,14} g^4_{k^+,k^+}$

Full expression not available (original memory size: 28.8 MB)

In[∘]:=
```
Module[{d = 2, es = {i, j, k, i⁺, j⁺, k⁺}, λ, λs},
  λs = λ_# & /@ Range[d - 1];
  Times[
    Normal@Series[Exp[r_d[j, k] + r_d[i, k⁺] + r_d[i⁺, j⁺]], {ϵ, 0, d}],
    Exp[Sum[g_{α,β} π_α ξ_β, {α, es}, {β, es}] + Sum[g_{α,β} π_α ξ_β + g_{β,α} π_β ξ_α, {α, es}, {β, λs}]]
   ] // Zip_{(p_#&/@es)⋃(x_#&/@es)} // Expand
 ] /. c_{1,_} → 0
```

Out[∘]=

$1+3\epsilon^2 c_{2,1}+\epsilon^2 c_{2,2} g_{i,i}+2\epsilon^2 c_{2,6} g^2_{i,i}+6\epsilon^2 c_{2,15} g^3_{i,i}+$ … 458 … $+$
$6\epsilon^2 c_{2,29} g_{k^+,i} g^2_{k^+,k^+}+6\epsilon^2 \pi_{\lambda\$64890_1} \xi_{\lambda\$64890_1} c_{2,29} g_{\lambda\$64890_1,i} g_{k^+,\lambda\$64890_1} g^2_{k^+,k^+}+$
$18\epsilon^2 \pi_{\lambda\$64890_1} \xi_{\lambda\$64890_1} c_{2,30} g_{\lambda\$64890_1,k^+} g_{k^+,\lambda\$64890_1} g^2_{k^+,k^+}+6\epsilon^2 c_{2,30} g^3_{k^+,k^+}$

Full expression not available (original memory size: 0.5 MB)

Non-Universally Solving at d=1

In[∘]:=
```
Short[lhs = Expand[Module[{d = 1, es = {i, j, k, i⁺, j⁺, k⁺}},
    Times[
      Normal@Series[Exp[r_d[j, k] + r_d[i, k⁺] + r_d[i⁺, j⁺]], {ϵ, 0, d}],
      Exp[Sum[g_{α,β} π_α ξ_β, {α, es}, {β, es}]]
     ] // Zip_{(p_#&/@es)⋃(x_#&/@es)} // Expand
   ] //. gRules_{1,j,k} ⋃ gRules_{1,i,k⁺} ⋃ gRules_{1,i⁺,j⁺}]]
```

Out[∘]//Short=

$1+3\epsilon c_{1,1}+\ll 119\gg+\frac{\epsilon\, c_{\ll 1\gg}\, g_{\ll 1\gg}\, g_{k^{\ll 4\gg},\ll 1\gg}}{T}$

In[ ]:=
```
Short[rhs = Expand[Module[{d = 1, es = {i, j, k, i⁺, j⁺, k⁺}},
    Times[
      Normal@Series[Exp[r_d[i, j] + r_d[i⁺, k] + r_d[j⁺, k⁺]], {ϵ, 0, d}],
      Exp[Sum[g_{α,β} π_α ξ_β, {α, es}, {β, es}]]
    ] // Zip_{(p_#&/@es)⋃(x_#&/@es)} // Expand
  ] //. gRules_{1,i,j} ⋃ gRules_{1,i⁺,k} ⋃ gRules_{1,j⁺,k⁺}]]
```

Out[ ]//Short=

$$1+3\,\epsilon\,c_{1,1}+\ll 119\gg+\frac{\epsilon\,c_{\ll 1\gg}\,g_{\ll 1\gg}\,g_{k^{\ll 4\gg},\ll 1\gg}}{T}$$

In[ ]:=
```
Exponent[lhs - rhs, T, Min]
```

Out[ ]=

∞

In[ ]:=
```
vars = Cases[Variables[lhs - rhs], c__]
```

Out[ ]=

{}

In[ ]:=
```
covars = DeleteCases[Variables[lhs - rhs], T | c__]
```

Out[ ]=

{}

In[ ]:=
```
Short[eqns = (# == 0) & /@ Union[Last /@ CoefficientRules[Expand[T^4 (lhs - rhs)], covars]]]
```

Out[ ]//Short=

{}

In[ ]:=
```
{sol} = Solve[eqns, vars]
```

Out[ ]=

{{}}

In[ ]:=
```
sol /. Rule → Set
```

Out[ ]=

{}

In[ ]:=
```
lhs = CF[(r_1[i, j] // px2g) //. gRules_{1,i,j} /.
    {ϵ → 1, c_{1,1} → -1/2, c_{1,2} → 1, c_{1,4} → -1, c_{1,9} → 1/2 (T - 1), c_{1,10} → -1}]
lhs == CF[R_1[1, i, j] //. gRules_{1,i,j}]
```

Out[ ]=

$$-\frac{1}{2}+g_{i^+,i^+}-g_{j^+,i^+}-\frac{(-1+T)\,g_{i^+,i^+}\,g_{j^+,i^+}}{T}-g_{i^+,j^+}\,g_{j^+,i^+}+\frac{(-1+T)\,g_{j^+,i^+}^2}{T}-g_{i^+,i^+}\,g_{j^+,j^+}+2\,g_{j^+,i^+}\,g_{j^+,j^+}$$

Out[ ]=

True

Solving at d=1

In[∘]:=
```
lhs = Expand[Module[{d = 1, es = {i, j, k, i⁺, j⁺, k⁺}, λs},
   λs = λ_# & /@ Range[d + 1];
   Times[
     Normal@Series[Exp[r_d[j, k] + r_d[i, k⁺] + r_d[i⁺, j⁺]], {ϵ, 0, d}],
     Exp[Sum[g_{α,β} π_α ξ_β, {α, es}, {β, es}] + Sum[g_{α,β} π_α ξ_β + g_{β,α} π_β ξ_α, {α, es}, {β, λs}]]
    ] // Zip_{(p_#&/@es)∪(x_#&/@es)} // Expand
  ] //. gRules_{1,j,k} ⋃ gRules_{1,i,k⁺} ⋃ gRules_{1,i⁺,j⁺}]
```

Out[∘]=

$1 + 3\,\epsilon\,c_{1,1} + 2\,\epsilon\,c_{1,2}\,g_{i^{++},i^{++}} - \epsilon\,c_{1,2}\,g_{j^{++},i^{++}} + \langle\langle 1874 \rangle\rangle + \frac{\epsilon\,\pi_{\lambda_2}^2\,\xi_{\lambda_2}^2\,c_{1,10}\,g_{k^{++},\lambda_2}^2\,g_{\lambda_2,j^{++}}\,g_{\lambda_2,k^{++}}}{2\,T}$

Full expression not available (original memory size: 1.7 MB)

In[∘]:=
```
rhs = Expand[Module[{d = 1, es = {i, j, k, i⁺, j⁺, k⁺}, λs},
   λs = λ_# & /@ Range[d + 1];
   Times[
     Normal@Series[Exp[r_d[i, j] + r_d[i⁺, k] + r_d[j⁺, k⁺]], {ϵ, 0, d}],
     Exp[Sum[g_{α,β} π_α ξ_β, {α, es}, {β, es}] + Sum[g_{α,β} π_α ξ_β + g_{β,α} π_β ξ_α, {α, es}, {β, λs}]]
    ] // Zip_{(p_#&/@es)∪(x_#&/@es)} // Expand
  ] //. gRules_{1,i,j} ⋃ gRules_{1,i⁺,k} ⋃ gRules_{1,j⁺,k⁺}]
```

Out[∘]=

$1 + 3\,\epsilon\,c_{1,1} + 2\,\epsilon\,c_{1,2}\,g_{i^{++},i^{++}} - \epsilon\,c_{1,2}\,g_{j^{++},i^{++}} + \langle\langle 1874 \rangle\rangle + \frac{\epsilon\,\pi_{\lambda_2}^2\,\xi_{\lambda_2}^2\,c_{1,10}\,g_{k^{++},\lambda_2}^2\,g_{\lambda_2,j^{++}}\,g_{\lambda_2,k^{++}}}{2\,T}$

Full expression not available (original memory size: 1.7 MB)

In[∘]:=
```
lhs - rhs
```

Out[∘]=
```
0
```

In[∘]:=
```
Exponent[lhs - rhs, T, Min]
```

Out[∘]=
```
∞
```

In[∘]:=
```
vars = Cases[Variables[lhs - rhs], c__]
```

Out[∘]=
```
{}
```

In[∘]:=
```
covars = DeleteCases[Variables[lhs - rhs], T | c__]
```

Out[∘]=
```
{}
```

In[∘]:=
```
Short[eqns = (# == 0) & /@ Union[Last /@ CoefficientRules[Expand[T^4 (lhs - rhs)], covars]]]
```

Out[∘]//Short=
```
{}
```

In[∘]:=
```
{sol} = Solve[eqns, vars]
```

In[∘]:=
```
sol /. Rule → Set
```

In[∘]:=
```
Simplify[r₁[j, k] /. {c_{1,1|2|4} → 0, c_{1,9} → 1, c_{1,13} → -1}]
```

In[∘]:=
```
c_{1,1} = c_{1,2} = c_{1,4} = 0; c_{1,9} = 1; c_{1,13} = -1;
r₁[j, k]
```

Non-Universally Solving at d=2

In[∘]:=
```
Short[lhs = Expand[Module[{d = 2, es = {i, j, k, i⁺, j⁺, k⁺}},
    Times[
      Normal@Series[Exp[r_d[j, k] + r_d[i, k⁺] + r_d[i⁺, j⁺]], {ϵ, 0, d}],
      Exp[Sum[g_{α,β} π_α ξ_β, {α, es}, {β, es}]]
     ] // Zip_{(p_#&/@es)⋃(x_#&/@es)} // Expand
   ] //. gRules_{1,j,k} ⋃ gRules_{1,i,k⁺} ⋃ gRules_{1,i⁺,j⁺}]]
```

Out[∘]//Short=

$$1 + 3\,\epsilon\, c_{1,1} + \frac{9}{2}\,\epsilon^2\, c_{1,1}^2 + \ll 8195 \gg + 12\,\epsilon^2\, c_{2,30}\, g_{k^{++},k^{++}}^3$$

In[∘]:=
```
Short[rhs = Expand[Module[{d = 2, es = {i, j, k, i⁺, j⁺, k⁺}},
    Times[
      Normal@Series[Exp[r_d[i, j] + r_d[i⁺, k] + r_d[j⁺, k⁺]], {ϵ, 0, d}],
      Exp[Sum[g_{α,β} π_α ξ_β, {α, es}, {β, es}]]
     ] // Zip_{(p_#&/@es)⋃(x_#&/@es)} // Expand
   ] //. gRules_{1,i,j} ⋃ gRules_{1,i⁺,k} ⋃ gRules_{1,j⁺,k⁺}]]
```

Out[∘]//Short=

$$1 + 3\,\epsilon\, c_{1,1} + \frac{9}{2}\,\epsilon^2\, c_{1,1}^2 + \ll 8135 \gg + 12\,\epsilon^2\, c_{2,30}\, g_{k^{++},k^{++}}^3$$

In[∘]:=
```
Exponent[lhs - rhs, T, Min]
```

Out[∘]=

-6

In[∘]:=
```
vars = Cases[Variables[lhs - rhs], c__]
```

Out[∘]=

$\{c_{1,2}, c_{1,4}, c_{1,9}, c_{1,10}, c_{2,2}, c_{2,3}, c_{2,4}, c_{2,5}, c_{2,6}, c_{2,7}, c_{2,8}, c_{2,9}, c_{2,10}, c_{2,11}, c_{2,12}, c_{2,13}, c_{2,14}, c_{2,15}, c_{2,16}, c_{2,17}, c_{2,18}, c_{2,19}, c_{2,20}, c_{2,21}, c_{2,22}, c_{2,23}, c_{2,24}, c_{2,25}, c_{2,26}, c_{2,27}, c_{2,28}, c_{2,29}, c_{2,30}\}$

In[∘]:=
```
covars = DeleteCases[Variables[lhs - rhs], T | c__]
```

Out[∘]=

$\{\epsilon, g_{i^{++},i^{++}}, g_{i^{++},j^{++}}, g_{i^{++},k^{++}}, g_{j^{++},i^{++}}, g_{j^{++},j^{++}}, g_{j^{++},k^{++}}, g_{k^{++},i^{++}}, g_{k^{++},j^{++}}, g_{k^{++},k^{++}}\}$

In[∘]:=
```
Short[eqns = (# == 0) & /@ Union[Last /@ CoefficientRules[Expand[T^6 (lhs - rhs)], covars]]]
```

Out[∘]//Short=

$$\{-T^5 c_{2,3} + T^6 c_{2,3} == 0,\ T^6 c_{2,3} - T^7 c_{2,3} == 0,\ \ll 125 \gg,\ \ll 1 \gg == 0,\ 12\,T^4 c_{1,9}^2 - 6\,T^4 c_{1,9} c_{1,10} + \ll 69 \gg + 36\,T^6 c_{2,30} == 0\}$$

In[ ]:=

```
{sol} = Solve[eqns, vars]
```

Solve: Equations may not give solutions for all "solve" variables.

Out[ ]=

$$\Big\{\Big\{c_{2,3}\to 0,\ c_{2,4}\to \frac{-c_{1,2}^2-c_{1,2}\,c_{1,4}-T\,c_{2,2}-T^2\,c_{2,5}}{T},\ c_{2,6}\to 0,\ c_{2,7}\to 0,\ c_{2,8}\to 0,$$

$$c_{2,9}\to \frac{1}{2\,(-1+T)\,T}\Big(2\,c_{1,2}\,c_{1,9}-2\,T^2\,c_{1,2}\,c_{1,9}+2\,c_{1,4}\,c_{1,9}-2\,T\,c_{1,4}\,c_{1,9}+2\,T\,c_{1,9}^2+c_{1,2}\,c_{1,10}-T\,c_{1,2}\,c_{1,10}+T^2\,c_{1,2}\,c_{1,10}-T^3\,c_{1,2}\,c_{1,10}+c_{1,4}\,c_{1,10}+T\,c_{1,4}\,c_{1,10}-2\,T^2\,c_{1,4}\,c_{1,10}-T\,c_{1,9}\,c_{1,10}+3\,T^2\,c_{1,9}\,c_{1,10}-T^2\,c_{1,10}^2+T^3\,c_{1,10}^2+T\,c_{2,10}-T^3\,c_{2,10}+2\,T\,c_{2,13}-2\,T^2\,c_{2,13}\Big),$$

$$c_{2,11}\to 0,\ c_{2,12}\to \frac{c_{1,2}\,c_{1,10}-T\,c_{1,2}\,c_{1,10}+c_{1,4}\,c_{1,10}+T\,c_{2,10}-T^2\,c_{2,10}+2\,T\,c_{2,13}-2\,T^2\,c_{2,13}}{2\,T},$$

$$c_{2,14}\to 0,\ c_{2,15}\to 0,\ c_{2,16}\to 0,\ c_{2,17}\to 0,\ c_{2,18}\to 0,$$

$$c_{2,19}\to \frac{-6\,c_{1,9}^2+3\,c_{1,9}\,c_{1,10}-9\,T\,c_{1,9}\,c_{1,10}-c_{1,10}^2+2\,T\,c_{1,10}^2-T^2\,c_{1,10}^2+6\,c_{2,29}-6\,T\,c_{2,29}}{6\,(-1+T)},$$

$$c_{2,20}\to -\frac{1}{2}\,c_{1,10}^2,\ c_{2,21}\to 0,\ c_{2,22}\to 0,$$

$$c_{2,23}\to \frac{1}{6}\Big(-12\,c_{1,9}^2+12\,c_{1,9}\,c_{1,10}-18\,T\,c_{1,9}\,c_{1,10}-c_{1,10}^2+8\,T\,c_{1,10}^2-7\,T^2\,c_{1,10}^2+6\,c_{2,29}-6\,T\,c_{2,29}\Big),$$

$$c_{2,24}\to \frac{2\,c_{1,9}^2-c_{1,9}\,c_{1,10}+3\,T\,c_{1,9}\,c_{1,10}-2\,c_{1,10}^2+2\,T^2\,c_{1,10}^2}{2\,(-1+T)},$$

$$c_{2,25}\to -\frac{1}{2}\,c_{1,10}^2,\ c_{2,26}\to 0,\ c_{2,27}\to \frac{1}{3}\,(-1+T)\,c_{1,10}\,(3\,c_{1,9}+2\,T\,c_{1,10}),$$

$$c_{2,28}\to \frac{1}{2}\Big(-2\,c_{1,9}\,c_{1,10}-c_{1,10}^2-T\,c_{1,10}^2+2\,c_{2,29}-2\,T\,c_{2,29}\Big),\ c_{2,30}\to 0\Big\}\Big\}$$

In[ ]:= **sol /. Rule → Set**

Out[ ]=

$$\left\{0, \frac{-c_{1,2}^2 - c_{1,2}\,c_{1,4} - T\,c_{2,2} - T^2\,c_{2,5}}{T}, 0, 0, 0, \frac{1}{2\,(-1+T)\,T}\left(2\,c_{1,2}\,c_{1,9} - 2\,T^2\,c_{1,2}\,c_{1,9} + 2\,c_{1,4}\,c_{1,9} - 2\,T\,c_{1,4}\,c_{1,9} + 2\,T\,c_{1,9}^2 + c_{1,2}\,c_{1,10} - T\,c_{1,2}\,c_{1,10} + T^2\,c_{1,2}\,c_{1,10} - T^3\,c_{1,2}\,c_{1,10} + c_{1,4}\,c_{1,10} + T\,c_{1,4}\,c_{1,10} - 2\,T^2\,c_{1,4}\,c_{1,10} - T\,c_{1,9}\,c_{1,10} + 3\,T^2\,c_{1,9}\,c_{1,10} - T^2\,c_{1,10}^2 + T^3\,c_{1,10}^2 + T\,c_{2,10} - T^3\,c_{2,10} + 2\,T\,c_{2,13} - 2\,T^2\,c_{2,13}\right),\right.$$

$$0, \frac{c_{1,2}\,c_{1,10} - T\,c_{1,2}\,c_{1,10} + c_{1,4}\,c_{1,10} + T\,c_{2,10} - T^2\,c_{2,10} + 2\,T\,c_{2,13} - 2\,T^2\,c_{2,13}}{2\,T}, 0, 0, 0, 0,$$

$$0, \frac{-6\,c_{1,9}^2 + 3\,c_{1,9}\,c_{1,10} - 9\,T\,c_{1,9}\,c_{1,10} - c_{1,10}^2 + 2\,T\,c_{1,10}^2 - T^2\,c_{1,10}^2 + 6\,c_{2,29} - 6\,T\,c_{2,29}}{6\,(-1+T)}, -\frac{1}{2}\,c_{1,10}^2,$$

$$0, 0, \frac{1}{6}\left(-12\,c_{1,9}^2 + 12\,c_{1,9}\,c_{1,10} - 18\,T\,c_{1,9}\,c_{1,10} - c_{1,10}^2 + 8\,T\,c_{1,10}^2 - 7\,T^2\,c_{1,10}^2 + 6\,c_{2,29} - 6\,T\,c_{2,29}\right),$$

$$\frac{2\,c_{1,9}^2 - c_{1,9}\,c_{1,10} + 3\,T\,c_{1,9}\,c_{1,10} - 2\,c_{1,10}^2 + 2\,T^2\,c_{1,10}^2}{2\,(-1+T)}, -\frac{1}{2}\,c_{1,10}^2, 0,$$

$$\left.\frac{1}{3}\,(-1+T)\,c_{1,10}\,(3\,c_{1,9} + 2\,T\,c_{1,10}), \frac{1}{2}\left(-2\,c_{1,9}\,c_{1,10} - c_{1,10}^2 - T\,c_{1,10}^2 + 2\,c_{2,29} - 2\,T\,c_{2,29}\right), 0\right\}$$

In[ ]:= **r_2[j, k]**

Out[ ]=

$$-\frac{\epsilon\,c_{1,1}}{-1+T} + \frac{T\,\epsilon\,c_{1,1}}{-1+T} - \frac{\epsilon\,p_j\,x_j\,c_{1,2}}{-1+T} + \frac{T\,\epsilon\,p_j\,x_j\,c_{1,2}}{-1+T} - \frac{\epsilon\,p_k\,x_k\,c_{1,2}}{-1+T} + \frac{\epsilon\,p_k\,x_k\,c_{1,2}}{(-1+T)\,T} - \frac{\epsilon^2\,p_k\,x_j\,c_{1,2}^2}{-1+T} +$$

$$\frac{\epsilon^2\,p_k\,x_j\,c_{1,2}^2}{(-1+T)\,T} - \frac{\epsilon\,p_k\,x_j\,c_{1,4}}{-1+T} + \frac{T\,\epsilon\,p_k\,x_j\,c_{1,4}}{-1+T} - \frac{\epsilon\,p_k\,x_k\,c_{1,4}}{-1+T} + \frac{\epsilon\,p_k\,x_k\,c_{1,4}}{(-1+T)\,T} - \frac{\epsilon^2\,p_k\,x_j\,c_{1,2}\,c_{1,4}}{-1+T} +$$

$$\frac{\epsilon^2\,p_k\,x_j\,c_{1,2}\,c_{1,4}}{(-1+T)\,T} - \frac{\epsilon\,p_j\,p_k\,x_j^2\,c_{1,9}}{-1+T} + \frac{T\,\epsilon\,p_j\,p_k\,x_j^2\,c_{1,9}}{-1+T} + \frac{\epsilon\,p_k^2\,x_j^2\,c_{1,9}}{-1+T} - \frac{2\,T\,\epsilon\,p_k^2\,x_j^2\,c_{1,9}}{-1+T} +$$

$$\frac{T^2\,\epsilon\,p_k^2\,x_j^2\,c_{1,9}}{-1+T} + \frac{\epsilon\,p_k^2\,x_j\,x_k\,c_{1,9}}{-1+T} - \frac{T\,\epsilon\,p_k^2\,x_j\,x_k\,c_{1,9}}{-1+T} + \frac{\epsilon^2\,p_j\,p_k\,x_j^2\,c_{1,2}\,c_{1,9}}{(-1+T)\,T} - \frac{T\,\epsilon^2\,p_j\,p_k\,x_j^2\,c_{1,2}\,c_{1,9}}{-1+T} -$$

$$\frac{\epsilon^2\,p_j\,p_k\,x_j^2\,c_{1,4}\,c_{1,9}}{-1+T} + \frac{\epsilon^2\,p_j\,p_k\,x_j^2\,c_{1,4}\,c_{1,9}}{(-1+T)\,T} + \frac{\epsilon^2\,p_j\,p_k\,x_j^2\,c_{1,9}^2}{-1+T} - \frac{\epsilon^2\,p_j^2\,p_k\,x_j^3\,c_{1,9}^2}{-1+T} + \frac{2\,\epsilon^2\,p_j\,p_k^2\,x_j^3\,c_{1,9}^2}{-1+T} -$$

$$\frac{2\,T\,\epsilon^2\,p_j\,p_k^2\,x_j^3\,c_{1,9}^2}{-1+T} + \frac{\epsilon^2\,p_j\,p_k^2\,x_j^2\,x_k\,c_{1,9}^2}{-1+T} + \frac{T\,\epsilon\,p_k^2\,x_j^2\,c_{1,10}}{2\,(-1+T)} - \frac{T^2\,\epsilon\,p_k^2\,x_j^2\,c_{1,10}}{-1+T} + \frac{T^3\,\epsilon\,p_k^2\,x_j^2\,c_{1,10}}{2\,(-1+T)} -$$

$$\frac{\epsilon\,p_j\,p_k\,x_j\,x_k\,c_{1,10}}{-1+T} + \frac{T\,\epsilon\,p_j\,p_k\,x_j\,x_k\,c_{1,10}}{-1+T} + \frac{\epsilon\,p_k^2\,x_j\,x_k\,c_{1,10}}{2\,(-1+T)} - \frac{T^2\,\epsilon\,p_k^2\,x_j\,x_k\,c_{1,10}}{2\,(-1+T)} - \frac{\epsilon^2\,p_j\,p_k\,x_j^2\,c_{1,2}\,c_{1,10}}{2\,(-1+T)} +$$

$$\frac{\epsilon^2\,p_j\,p_k\,x_j^2\,c_{1,2}\,c_{1,10}}{2\,(-1+T)\,T} + \frac{T\,\epsilon^2\,p_j\,p_k\,x_j^2\,c_{1,2}\,c_{1,10}}{2\,(-1+T)} - \frac{T^2\,\epsilon^2\,p_j\,p_k\,x_j^2\,c_{1,2}\,c_{1,10}}{2\,(-1+T)} + \frac{\epsilon^2\,p_k^2\,x_j^2\,c_{1,2}\,c_{1,10}}{-1+T} -$$

$$\frac{\epsilon^2\,p_k^2\,x_j^2\,c_{1,2}\,c_{1,10}}{2\,(-1+T)\,T} - \frac{T\,\epsilon^2\,p_k^2\,x_j^2\,c_{1,2}\,c_{1,10}}{2\,(-1+T)} + \frac{\epsilon^2\,p_j\,p_k\,x_j^2\,c_{1,4}\,c_{1,10}}{2\,(-1+T)} + \frac{\epsilon^2\,p_j\,p_k\,x_j^2\,c_{1,4}\,c_{1,10}}{2\,(-1+T)\,T} -$$

$$\frac{T \epsilon^2 p_j p_k x_j^2 c_{1,4} c_{1,10}}{-1+T} + \frac{\epsilon^2 p_k^2 x_j^2 c_{1,4} c_{1,10}}{2(-1+T)} - \frac{\epsilon^2 p_k^2 x_j^2 c_{1,4} c_{1,10}}{2(-1+T)T} - \frac{\epsilon^2 p_j p_k x_j^2 c_{1,9} c_{1,10}}{2(-1+T)} +$$

$$\frac{3T\epsilon^2 p_j p_k x_j^2 c_{1,9}c_{1,10}}{2(-1+T)} + \frac{\epsilon^2 p_j^2 p_k x_j^3 c_{1,9} c_{1,10}}{2(-1+T)} - \frac{3T\epsilon^2 p_j^2 p_k x_j^3 c_{1,9}c_{1,10}}{2(-1+T)} - \frac{2\epsilon^2 p_j p_k^2 x_j^3 c_{1,9}c_{1,10}}{-1+T} +$$

$$\frac{5T\epsilon^2 p_j p_k^2 x_j^3 c_{1,9}c_{1,10}}{-1+T} - \frac{3T^2\epsilon^2 p_j p_k^2 x_j^3 c_{1,9}c_{1,10}}{-1+T} + \frac{\epsilon^2 p_k^3 x_j^3 c_{1,9}c_{1,10}}{-1+T} - \frac{2T\epsilon^2 p_k^3 x_j^3 c_{1,9}c_{1,10}}{-1+T} +$$

$$\frac{T^2\epsilon^2 p_k^3 x_j^3 c_{1,9}c_{1,10}}{-1+T} - \frac{\epsilon^2 p_j p_k^2 x_j^2 x_k c_{1,9}c_{1,10}}{2(-1+T)} + \frac{3T\epsilon^2 p_j p_k^2 x_j^2 x_k c_{1,9}c_{1,10}}{2(-1+T)} + \frac{\epsilon^2 p_k^3 x_j^2 x_k c_{1,9}c_{1,10}}{-1+T} -$$

$$\frac{T\epsilon^2 p_k^3 x_j^2 x_k c_{1,9}c_{1,10}}{-1+T} - \frac{T\epsilon^2 p_j p_k x_j^2 c_{1,10}^2}{2(-1+T)} + \frac{T^2\epsilon^2 p_j p_k x_j^2 c_{1,10}^2}{2(-1+T)} - \frac{\epsilon^2 p_j^2 p_k x_j^3 c_{1,10}^2}{6(-1+T)} + \frac{T\epsilon^2 p_j^2 p_k x_j^3 c_{1,10}^2}{3(-1+T)} -$$

$$\frac{T^2\epsilon^2 p_j^2 p_k x_j^3 c_{1,10}^2}{6(-1+T)} + \frac{\epsilon^2 p_j p_k^2 x_j^3 c_{1,10}^2}{6(-1+T)} - \frac{3T\epsilon^2 p_j p_k^2 x_j^3 c_{1,10}^2}{2(-1+T)} + \frac{5T^2\epsilon^2 p_j p_k^2 x_j^3 c_{1,10}^2}{2(-1+T)} -$$

$$\frac{7T^3\epsilon^2 p_j p_k^2 x_j^3 c_{1,10}^2}{6(-1+T)} + \frac{2T\epsilon^2 p_k^3 x_j^3 c_{1,10}^2}{3(-1+T)} - \frac{4T^2\epsilon^2 p_k^3 x_j^3 c_{1,10}^2}{3(-1+T)} + \frac{2T^3\epsilon^2 p_k^3 x_j^3 c_{1,10}^2}{3(-1+T)} + \frac{\epsilon^2 p_j^2 p_k x_j^2 x_k c_{1,10}^2}{2(-1+T)} -$$

$$\frac{T\epsilon^2 p_j^2 p_k x_j^2 x_k c_{1,10}^2}{2(-1+T)} - \frac{\epsilon^2 p_j p_k^2 x_j^2 x_k c_{1,10}^2}{-1+T} + \frac{T^2\epsilon^2 p_j p_k^2 x_j^2 x_k c_{1,10}^2}{-1+T} + \frac{\epsilon^2 p_k^3 x_j^2 x_k c_{1,10}^2}{2(-1+T)} -$$

$$\frac{T^2\epsilon^2 p_k^3 x_j^2 x_k c_{1,10}^2}{2(-1+T)} + \frac{\epsilon^2 p_j p_k^2 x_j x_k^2 c_{1,10}^2}{2(-1+T)} - \frac{T\epsilon^2 p_j p_k^2 x_j x_k^2 c_{1,10}^2}{2(-1+T)} - \frac{\epsilon^2 c_{2,1}}{-1+T} + \frac{T\epsilon^2 c_{2,1}}{-1+T} - \frac{\epsilon^2 p_j x_j c_{2,2}}{-1+T} +$$

$$\frac{T\epsilon^2 p_j x_j c_{2,2}}{-1+T} + \frac{\epsilon^2 p_k x_j c_{2,2}}{-1+T} - \frac{T\epsilon^2 p_k x_j c_{2,2}}{-1+T} + \frac{T\epsilon^2 p_k x_j c_{2,5}}{-1+T} - \frac{T^2\epsilon^2 p_k x_j c_{2,5}}{-1+T} - \frac{\epsilon^2 p_k x_k c_{2,5}}{-1+T} +$$

$$\frac{T\epsilon^2 p_k x_k c_{2,5}}{-1+T} + \frac{\epsilon^2 p_j p_k x_j^2 c_{2,10}}{2(-1+T)} - \frac{T^2\epsilon^2 p_j p_k x_j^2 c_{2,10}}{2(-1+T)} - \frac{\epsilon^2 p_k^2 x_j^2 c_{2,10}}{2(-1+T)} + \frac{T\epsilon^2 p_k^2 x_j^2 c_{2,10}}{-1+T} -$$

$$\frac{T^2\epsilon^2 p_k^2 x_j^2 c_{2,10}}{2(-1+T)} - \frac{\epsilon^2 p_j p_k x_j x_k c_{2,10}}{-1+T} + \frac{T\epsilon^2 p_j p_k x_j x_k c_{2,10}}{-1+T} + \frac{\epsilon^2 p_j p_k x_j^2 c_{2,13}}{-1+T} - \frac{T\epsilon^2 p_j p_k x_j^2 c_{2,13}}{-1+T} -$$

$$\frac{\epsilon^2 p_k^2 x_j^2 c_{2,13}}{-1+T} + \frac{2T\epsilon^2 p_k^2 x_j^2 c_{2,13}}{-1+T} - \frac{T^2\epsilon^2 p_k^2 x_j^2 c_{2,13}}{-1+T} - \frac{\epsilon^2 p_k^2 x_j x_k c_{2,13}}{-1+T} + \frac{T\epsilon^2 p_k^2 x_j x_k c_{2,13}}{-1+T} +$$

$$\frac{\epsilon^2 p_j^2 p_k x_j^3 c_{2,29}}{-1+T} - \frac{T\epsilon^2 p_j^2 p_k x_j^3 c_{2,29}}{-1+T} - \frac{\epsilon^2 p_j p_k^2 x_j^3 c_{2,29}}{-1+T} + \frac{2T\epsilon^2 p_j p_k^2 x_j^3 c_{2,29}}{-1+T} - \frac{T^2\epsilon^2 p_j p_k^2 x_j^3 c_{2,29}}{-1+T} -$$

$$\frac{\epsilon^2 p_k^3 x_j^2 x_k c_{2,29}}{-1+T} + \frac{2T\epsilon^2 p_k^3 x_j^2 x_k c_{2,29}}{-1+T} - \frac{T^2\epsilon^2 p_k^3 x_j^2 x_k c_{2,29}}{-1+T} - \frac{\epsilon^2 p_k^3 x_j x_k^2 c_{2,29}}{-1+T} + \frac{T\epsilon^2 p_k^3 x_j x_k^2 c_{2,29}}{-1+T}$$

Solving at d=2

In[ ]:= $r_2[j, k]$

Out[∘]=

$$\in p_j p_k x_j^2 - \in p_k^2 x_j^2 + T \in p_k^2 x_j^2 - \in p_k^2 x_j x_k + \in^2 c_{2,1} + \in^2 p_j x_j c_{2,2} + \in^2 p_j x_k c_{2,3} + \in^2 p_k x_j c_{2,4} + \in^2 p_k x_k c_{2,5} + \in^2 p_j^2 x_j^2 c_{2,6} + \in^2 p_j^2 x_j x_k c_{2,7} + \in^2 p_j^2 x_k^2 c_{2,8} + \in^2 p_j p_k x_j^2 c_{2,9} + \in^2 p_j p_k x_j x_k c_{2,10} + \in^2 p_j p_k x_k^2 c_{2,11} + \in^2 p_k^2 x_j^2 c_{2,12} + \in^2 p_k^2 x_j x_k c_{2,13} + \in^2 p_k^2 x_k^2 c_{2,14} + \in^2 p_j^3 x_j^3 c_{2,15} + \in^2 p_j^3 x_j^2 x_k c_{2,16} + \in^2 p_j^3 x_j x_k^2 c_{2,17} + \in^2 p_j^3 x_k^3 c_{2,18} + \in^2 p_j^2 p_k x_j^3 c_{2,19} + \in^2 p_j^2 p_k x_j^2 x_k c_{2,20} + \in^2 p_j^2 p_k x_j x_k^2 c_{2,21} + \in^2 p_j^2 p_k x_k^3 c_{2,22} + \in^2 p_j p_k^2 x_j^3 c_{2,23} + \in^2 p_j p_k^2 x_j^2 x_k c_{2,24} + \in^2 p_j p_k^2 x_j x_k^2 c_{2,25} + \in^2 p_j p_k^2 x_k^3 c_{2,26} + \in^2 p_k^3 x_j^3 c_{2,27} + \in^2 p_k^3 x_j^2 x_k c_{2,28} + \in^2 p_k^3 x_j x_k^2 c_{2,29} + \in^2 p_k^3 x_k^3 c_{2,30}$$

In[∘]:=

```
lhs = Expand[Module[{d = 2, es = {i, j, k, i⁺, j⁺, k⁺}, λs},
    λs = λ_# & /@ Range[d + 1];
    Times[
      Normal@Series[Exp[r_d[j, k] + r_d[i, k⁺] + r_d[i⁺, j⁺]], {ϵ, 0, d}],
      Exp[Sum[g_{α,β} π_α ξ_β, {α, es}, {β, es}] + Sum[g_{α,β} π_α ξ_β + g_{β,α} π_β ξ_α, {α, es}, {β, λs}]]
     ] // Zip_{(p_#&/@es)⋃(x_#&/@es)} // Expand
   ] //. gRules_{1,j,k} ⋃ gRules_{1,i,k⁺} ⋃ gRules_{1,i⁺,j⁺}]
```

Out[∘]=

$$1 + 3 \in c_{1,1} + \frac{9}{2} \in^2 c_{1,1}^2 - \frac{3 \in^2 c_{2,1}}{-1+T} + \cdots 2874844 \cdots + \frac{9}{8} \in^2 \pi_{\lambda_3}^4 \xi_{\lambda_3}^4 c_{1,10}^2 g_{k^{++},\lambda_3}^4 g_{\lambda_3,j^{++}}^2 g_{\lambda_3,k^{++}}^2 + \frac{\in^2 \pi_{\lambda_3}^4 \xi_{\lambda_3}^4 c_{1,10}^2 g_{k^{++},\lambda_3}^4 g_{\lambda_3,j^{++}}^2 g_{\lambda_3,k^{++}}^2}{8 T^2} - \frac{3 \in^2 \pi_{\lambda_3}^4 \xi_{\lambda_3}^4 c_{1,10}^2 g_{k^{++},\lambda_3}^4 g_{\lambda_3,j^{++}}^2 g_{\lambda_3,k^{++}}^2}{4 T}$$

Full expression not available (original memory size: 4593.4 MB)

In[∘]:=

```
rhs = Expand[Module[{d = 2, es = {i, j, k, i⁺, j⁺, k⁺}, λs},
    λs = λ_# & /@ Range[d + 1];
    Times[
      Normal@Series[Exp[r_d[i, j] + r_d[i⁺, k] + r_d[j⁺, k⁺]], {ϵ, 0, d}],
      Exp[Sum[g_{α,β} π_α ξ_β, {α, es}, {β, es}] + Sum[g_{α,β} π_α ξ_β + g_{β,α} π_β ξ_α, {α, es}, {β, λs}]]
     ] // Zip_{(p_#&/@es)⋃(x_#&/@es)} // Expand
   ] //. gRules_{1,i,j} ⋃ gRules_{1,i⁺,k} ⋃ gRules_{1,j⁺,k⁺}]
```

Out[∘]=

$$1 + 3 \in c_{1,1} + \frac{9}{2} \in^2 c_{1,1}^2 - \frac{3 \in^2 c_{2,1}}{-1+T} + \cdots 2844396 \cdots + \frac{9}{8} \in^2 \pi_{\lambda_3}^4 \xi_{\lambda_3}^4 c_{1,10}^2 g_{k^{++},\lambda_3}^4 g_{\lambda_3,j^{++}}^2 g_{\lambda_3,k^{++}}^2 + \frac{\in^2 \pi_{\lambda_3}^4 \xi_{\lambda_3}^4 c_{1,10}^2 g_{k^{++},\lambda_3}^4 g_{\lambda_3,j^{++}}^2 g_{\lambda_3,k^{++}}^2}{8 T^2} - \frac{3 \in^2 \pi_{\lambda_3}^4 \xi_{\lambda_3}^4 c_{1,10}^2 g_{k^{++},\lambda_3}^4 g_{\lambda_3,j^{++}}^2 g_{\lambda_3,k^{++}}^2}{4 T}$$

Full expression not available (original memory size: 4551.3 MB)

In[∘]:=

```
Exponent[lhs - rhs, T, Min]
```

Out[∘]=

-6

In[∘]:=

```
vars = Cases[Variables[lhs - rhs], c__]
```

Out[∘]=

$\{c_{1,2}, c_{1,4}, c_{1,9}, c_{1,10}\}$

In[∘]:=
```
covars = DeleteCases[Variables[lhs - rhs], T | c__]
```

Out[∘]=

$$\{\epsilon, \pi_{\lambda_1}, \pi_{\lambda_2}, \pi_{\lambda_3}, \xi_{\lambda_1}, \xi_{\lambda_2}, \xi_{\lambda_3}, g_{i^{++},i^{++}}, g_{i^{++},j^{++}}, g_{i^{++},k^{++}}, g_{i^{++},\lambda_1}, g_{i^{++},\lambda_2}, g_{i^{++},\lambda_3}, g_{j^{++},i^{++}}, g_{j^{++},j^{++}}, g_{j^{++},k^{++}}, g_{j^{++},\lambda_1}, g_{j^{++},\lambda_2}, g_{j^{++},\lambda_3}, g_{k^{++},i^{++}}, g_{k^{++},j^{++}}, g_{k^{++},k^{++}}, g_{k^{++},\lambda_1}, g_{k^{++},\lambda_2}, g_{k^{++},\lambda_3}, g_{\lambda_1,i^{++}}, g_{\lambda_1,j^{++}}, g_{\lambda_1,k^{++}}, g_{\lambda_2,i^{++}}, g_{\lambda_2,j^{++}}, g_{\lambda_2,k^{++}}, g_{\lambda_3,i^{++}}, g_{\lambda_3,j^{++}}, g_{\lambda_3,k^{++}}\}$$

In[∘]:=
```
Short[eqns = (# == 0) & /@ Union[Last /@ CoefficientRules[Expand[T^6 (lhs - rhs)], covars]]]
```

Out[∘]//Short=

$$\left\{\ll 1 \gg, \ll 1 \gg == 0, \ll 82 \gg, -24\, T^3\, c_{1,9}^2 + \ll 25 \gg + \frac{10 \ll 1 \gg \ll 1 \gg \ll 1 \gg}{1 - T} == 0, -12\, T^3\, c_{1,9}^2 + \frac{12\, T^3\, c_{1,9}^2}{1 - T} + \ll 27 \gg + \frac{12\, T^7\, c_{1,10}^2}{1 - T} == 0\right\}$$

In[∘]:=
```
Simplify[eqns]
```

Out[∘]=

{True, True, True, True, True, True, True, True, True, True, True, True, True, True, True,
True, True, True, True, True, True, True, True, True, True, True, True, True, True, True,
True, True, True, True, True, True, True, True, True, True, True, True, True, True,
True, True, True, True, True, True, True, True, True, True, True, True, True, True,
True, True, True, True, True, True, True, True, True, True, True, True, True, True,
True, True, True, True, True, True, True, True, True, True, True, True, True, True}

Non-Universally Solving at d=3

In[∘]:=
```
Short[lhs = Expand[Module[{d = 3, es = {i, j, k, i⁺, j⁺, k⁺}},
    Times[
      Normal@Series[Exp[r_d[j, k] + r_d[i, k⁺] + r_d[i⁺, j⁺]], {ϵ, 0, d}],
      Exp[Sum[g_{α,β} π_α ξ_β, {α, es}, {β, es}]]
    ] // Zip_{(p_#&/@es)⋃(x_#&/@es)} // Expand
  ] //. gRules_{1,j,k} ⋃ gRules_{1,i,k⁺} ⋃ gRules_{1,i⁺,j⁺}]]
```

Out[∘]//Short=

$$1 + 3\,\epsilon\, c_{1,1} + \frac{9}{2}\,\epsilon^2\, c_{1,1}^2 + \ll 124670 \gg + \frac{48\, T\, \epsilon^3\, c_{3,55}\, g_{k^{++},\ll 1 \gg}^4}{-1 + T}$$

In[∘]:=
```
Short[rhs = Expand[Module[{d = 3, es = {i, j, k, i⁺, j⁺, k⁺}},
    Times[
      Normal@Series[Exp[r_d[i, j] + r_d[i⁺, k] + r_d[j⁺, k⁺]], {ϵ, 0, d}],
      Exp[Sum[g_{α,β} π_α ξ_β, {α, es}, {β, es}]]
    ] // Zip_{(p_#&/@es)⋃(x_#&/@es)} // Expand
  ] //. gRules_{1,i,j} ⋃ gRules_{1,i⁺,k} ⋃ gRules_{1,j⁺,k⁺}]]
```

Out[∘]//Short=

$$1 + 3\,\epsilon\, c_{1,1} + \frac{9}{2}\,\epsilon^2\, c_{1,1}^2 + \ll 121119 \gg + \frac{48\, T\, \epsilon^3\, c_{3,55}\, g_{k^{++},\ll 1 \gg}^4}{-1 + T}$$

In[ ]:=
```
Exponent[lhs - rhs, T, Min]
```
Out[ ]=
-10

In[ ]:=
```
vars = Cases[Variables[lhs - rhs], c__]
```
Out[ ]=
$\{c_{1,1}, c_{1,2}, c_{1,4}, c_{1,9}, c_{1,10}, c_{2,2}, c_{2,5}, c_{2,10}, c_{2,13}, c_{2,29}, c_{3,2}, c_{3,3}, c_{3,4}, c_{3,5}, c_{3,6}, c_{3,7}, c_{3,8}, c_{3,9}, c_{3,10}, c_{3,11}, c_{3,12}, c_{3,13}, c_{3,14}, c_{3,15}, c_{3,16}, c_{3,17}, c_{3,18}, c_{3,19}, c_{3,20}, c_{3,21}, c_{3,22}, c_{3,23}, c_{3,24}, c_{3,25}, c_{3,26}, c_{3,27}, c_{3,28}, c_{3,29}, c_{3,30}, c_{3,31}, c_{3,32}, c_{3,33}, c_{3,34}, c_{3,35}, c_{3,36}, c_{3,37}, c_{3,38}, c_{3,39}, c_{3,40}, c_{3,41}, c_{3,42}, c_{3,43}, c_{3,44}, c_{3,45}, c_{3,46}, c_{3,47}, c_{3,48}, c_{3,49}, c_{3,50}, c_{3,51}, c_{3,52}, c_{3,53}, c_{3,54}, c_{3,55}\}$

In[ ]:=
```
covars = DeleteCases[Variables[lhs - rhs], T | c__]
```
Out[ ]=
$\{\in, g_{i^{++},i^{++}}, g_{i^{++},j^{++}}, g_{i^{++},k^{++}}, g_{j^{++},i^{++}}, g_{j^{++},j^{++}}, g_{j^{++},k^{++}}, g_{k^{++},i^{++}}, g_{k^{++},j^{++}}, g_{k^{++},k^{++}}\}$

In[ ]:=
```
Short[eqns =
  (# == 0) & /@ Union@Simplify[Last /@ CoefficientRules[Expand[T^10 (lhs - rhs)], covars]]]
```
Out[ ]//Short=
$\{\text{True}, \ll 347 \gg, T^6 \left(72 (-1+T) c_{1,9}^3 + 12 \ll 2 \gg c_{\ll 1 \gg} + \ll 1 \gg + (-1+T) \left(\left(-29+109 T-146 T^2+48 T^3\right) c_{1,10}^3 + \ll 1 \gg - 6 (\ll 1 \gg)\right)\right) == 0\}$

In[ ]:=
```
{sol3} = Solve[eqns, vars]
```
Out[ ]=
$Aborted

```
sol3 /. Rule → Set
```

```
r3[j, k]
```