Pensieve header: Developing ρ_d.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\Oaxaca-2210"];
```

In[•]:=
```
Once[<< KnotTheory`; << Rot.m];
```

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.

Loading Rot.m from http://drorbn.net/la22/ap to compute rotation numbers.
```

```
<< "../../Projects/Profile/Profile.m"
```

```
This is Profile.m of http://www.drorbn.net/AcademicPensieve/Projects/Profile/.

This version: April 2020. Original version: July 1994.
```

In[•]:=
```
CF[ℰ_] := Module[{vs = Union[{ϵ}, Cases[ℰ, (g | p | x)__, ∞]]}, Total[
   CoefficientRules[Expand@ℰ, vs] /. (ps_ → c_) :> Factor[c] (Times @@ vs^ps)
  ]]
```

In[•]:=
```
{p^*, x^*, π^*, ξ^*} = {π, ξ, p, x};  (u_i_)^* := (u^*)_i;
```

In[•]:=
```
Zip_{}[ℰ_] := ℰ;
Zip_{ξ_,ξs___}[ℰ_] := (Collect[ℰ // Zip_{ξs}, ξ] /. f_. ξ^d_. :> (D[f, {ξ^*, d}])) /. ξ^* → 0
```

In[•]:=
```
γ1[1, k_] := ϵ (1/2 - p_k x_k); γ1[-1, k_] := ϵ (-1/2 + p_k x_k);
```

In[•]:=
```
r1[1, i_, j_] := ϵ (-1/2 + p_i x_i - p_j x_i + 1/2 (-1 + T) p_i p_j x_i^2 + 1/2 (1 - T) p_j^2 x_i^2 - p_i p_j x_i x_j + p_j^2 x_i x_j);
r1[-1, i_, j_] := ϵ (1/2 - p_i x_i + p_j x_i + ((-1 + T) p_i p_j x_i^2)/(2 T) - ((-1 + T) p_j^2 x_i^2)/(2 T) + p_i p_j x_i x_j - p_j^2 x_i x_j);
```

In[•]:=
```
γ2[1, k_] := -1/2 ϵ^2 p_k x_k + ϵ (1/2 - p_k x_k); γ2[-1, k_] := -1/2 ϵ^2 p_k x_k + ϵ (-1/2 + p_k x_k);
```

In[ ]:=
$$r_2[1,i\_,j\_] := \epsilon\left(-\frac{1}{2}+p_i x_i - p_j x_i + \frac{1}{2}(-1+T)\,p_i p_j x_i^2 + \frac{1}{2}(1-T)\,p_j^2 x_i^2 - p_i p_j x_i x_j + p_j^2 x_i x_j\right) +$$
$$\epsilon^2\left(-\frac{1}{2}p_i x_i + \frac{p_j x_i}{2} + \frac{1}{4}(1-3T)\,p_i p_j x_i^2 + \frac{1}{4}(-1+3T)\,p_j^2 x_i^2 + \frac{1}{3}(-1+T)\,p_i^2 p_j x_i^3 - \right.$$
$$\frac{1}{6}(-1+T)(5+T)\,p_i p_j^2 x_i^3 + \frac{1}{6}(-1+T)(3+T)\,p_j^3 x_i^3 + \frac{3}{2}p_i p_j x_i x_j - \frac{3}{2}p_j^2 x_i x_j -$$
$$\left.\frac{1}{2}p_i^2 p_j x_i^2 x_j + \frac{1}{2}(2+T)\,p_i p_j^2 x_i^2 x_j + \frac{1}{2}(-1-T)\,p_j^3 x_i^2 x_j - \frac{1}{2}p_i p_j^2 x_i x_j^2 + \frac{1}{2}p_j^3 x_i x_j^2\right);$$

In[ ]:=
$$r_2[-1,i\_,j\_] := \epsilon\left(\frac{1}{2} - p_i x_i + p_j x_i + \frac{(-1+T)\,p_i p_j x_i^2}{2T} - \frac{(-1+T)\,p_j^2 x_i^2}{2T} + p_i p_j x_i x_j - p_j^2 x_i x_j\right) +$$
$$\epsilon^2\left(-\frac{1}{2}p_i x_i + \frac{p_j x_i}{2} + \frac{(-3+T)\,p_i p_j x_i^2}{4T} - \frac{(-3+T)\,p_j^2 x_i^2}{4T} - \frac{(-1+T)\,p_i^2 p_j x_i^3}{3T} + \right.$$
$$\frac{(-1+T)(1+5T)\,p_i p_j^2 x_i^3}{6T^2} - \frac{(-1+T)(1+3T)\,p_j^3 x_i^3}{6T^2} + \frac{3}{2}p_i p_j x_i x_j - \frac{3}{2}p_j^2 x_i x_j -$$
$$\left.\frac{1}{2}p_i^2 p_j x_i^2 x_j + \frac{(1+2T)\,p_i p_j^2 x_i^2 x_j}{2T} - \frac{(1+T)\,p_j^3 x_i^2 x_j}{2T} - \frac{1}{2}p_i p_j^2 x_i x_j^2 + \frac{1}{2}p_j^3 x_i x_j^2\right);$$

In[ ]:=
```
Module[{k, x1, x2, p1, p2},
  γd_[0, j_] := 0;
  {x1*, x2*, p1*, p2*} = {p1, p2, x1, x2};
  rd_[s_, φi_, φj_, i_, j_] := Normal[Log[O[ϵ]^(d+1) + Zip{x1,x2}[Exp[O[ϵ]^(d+1) +
        (γd[φi, i] /. xi → xi + x1) +
        (γd[φj, j] /. xj → xj + x2) + (rd[s, i, j] /. {pi → pi - p1, pj → pj - p2})
      ]]]]
 ];
```

In[•]:=
```
ρd_[K_] := PPρd@Module[{Cs, φ, n, A, s, i, j, k, Δ, G, ρd},
  PP"Green"[
   {Cs, φ} = Rot[K]; n = Length[Cs];
   A = IdentityMatrix[2 n + 1];
   Cases[Cs, {s_, i_, j_} :> (A[[{i, j}, {i + 1, j + 1}]] += ({{-T^s, T^s - 1}, {0, -1}}))];
   Δ = T^((-Total[φ] - Total[Cs[[All, 1]]])/2) Det[A];
   G = Inverse[A];
  ];
  PPPairing[
   ρd = Times[
     Series[Exp[Total@Cases[Cs, {s_, i_, j_} :> rd[s, φ[[i]], φ[[j]], i, j]]],
       {ϵ, 0, d}] // Normal,
     Exp[Sum[gα,β πα ξβ, {α, 2 n + 1}, {β, 2 n + 1}]]
    ] // ZipJoin@@Table[{pk, xk}, {k, 2 n + 1}];
  ];
  PPRenormalizing[
   ρd = CoefficientList[Δ Normal[Series[ρd, {ϵ, 0, d}]] /. ϵ → Δ ϵ, ϵ, d + 1]];
  PPSubstitution@Factor[ρd /. α_^+ :> α + 1 /. gα_,β_ :> G[[α, β]]]
 ];
```

In[•]:=
```
{ca1,2 = 1, ca1,10 = -1, ca2,1 = 0, cb2,10 = 3 / 2};
```

In[•]:=
```
Table[K → ρ1[K], {K, AllKnots[{3, 6}]}]
```

KnotTheory: Loading precomputed data in PD4Knots`.

Out[•]=

$$\left\{\text{Knot}[3,1]\to\left\{\frac{1-T+T^2}{T},\ \frac{(-1+T)^2\left(1+T^2\right)}{T^2}\right\},\ \text{Knot}[4,1]\to\left\{-\frac{1-3T+T^2}{T},\ 0\right\},\right.$$

$$\text{Knot}[5,1]\to\left\{\frac{1-T+T^2-T^3+T^4}{T^2},\ \frac{(-1+T)^2\left(1+T^2\right)\left(2+T^2+2T^4\right)}{T^4}\right\},$$

$$\text{Knot}[5,2]\to\left\{\frac{2-3T+2T^2}{T},\ \frac{(-1+T)^2\left(5-4T+5T^2\right)}{T^2}\right\},$$

$$\text{Knot}[6,1]\to\left\{-\frac{(-2+T)(-1+2T)}{T},\ \frac{(-1+T)^2\left(1-4T+T^2\right)}{T^2}\right\},$$

$$\text{Knot}[6,2]\to\left\{-\frac{1-3T+3T^2-3T^3+T^4}{T^2},\ \frac{(-1+T)^2\left(1-4T+4T^2-4T^3+4T^4-4T^5+T^6\right)}{T^4}\right\},$$

$$\left.\text{Knot}[6,3]\to\left\{\frac{1-3T+5T^2-3T^3+T^4}{T^2},\ 0\right\}\right\}$$

In[•]:=
```
ρ2[Knot[3, 1]]
```

Out[•]=

$$\left\{\frac{1-T+T^2}{T},\ \frac{(-1+T)^2\left(1+T^2\right)}{T^2},\ \frac{1-4T+7T^2-12T^3+18T^4-12T^5+7T^6-4T^7+T^8}{2T^3\left(1-T+T^2\right)}\right\}$$

In[•]:=
```
BeginProfile[]
Timing[z1 = ρ2[Knot[10, 106]]]
PrintProfile[]
```

Out[•]=

ProfileRoot

Out[•]=

$$\Big\{173.219,$$
$$\Big\{-\frac{\left(1-T+T^2\right)\left(-1+T-2T^2+T^3\right)\left(-1+2T-T^2+T^3\right)}{T^4},\ -\frac{1}{T^8}(-1+T)^2\left(1-6T+20T^2-48T^3+82T^4-114T^5+134T^6-140T^7+134T^8-114T^9+82T^{10}-48T^{11}+20T^{12}-6T^{13}+T^{14}\right),$$
$$-\Big(\left(1-16T+127T^2-676T^3+2735T^4-8980T^5+24\,938T^6-60\,420T^7+131\,072T^8-259\,992T^9+477\,614T^{10}-814\,576T^{11}+1\,282\,448T^{12}-1\,846\,716T^{13}+2\,411\,126T^{14}-2\,836\,312T^{15}+2\,995\,252T^{16}-2\,836\,312T^{17}+2\,411\,126T^{18}-1\,846\,716T^{19}+1\,282\,448T^{20}-814\,576T^{21}+477\,614T^{22}-259\,992T^{23}+131\,072T^{24}-60\,420T^{25}+24\,938T^{26}-8980T^{27}+2735T^{28}-676T^{29}+127T^{30}-16T^{31}+T^{32}\right)\Big/\left(2T^{12}\left(1-T+T^2\right)\left(-1+T-2T^2+T^3\right)\left(-1+2T-T^2+T^3\right)\right)\Big)\Big\}\Big\}$$

Out[•]=
```
ProfileRoot is root. Profiled time: 173.219
  ( 1)      0/ 173.220 above ρd
Pairing: called 1 times, time in 156.422/156.422
  ( 1)  156.420/ 156.420 under ρd
Renormalizing: called 1 times, time in 15.859/15.859
  ( 1)  15.859/ 15.859 under ρd
Green: called 1 times, time in 0.547/0.547
  ( 1)   0.547/  0.547 under ρd
Substitution: called 1 times, time in 0.391/0.391
  ( 1)   0.391/  0.391 under ρd
ρd: called 1 times, time in 0./173.219
  ( 1)      0/ 173.220 under ProfileRoot
  ( 1)   0.547/  0.547 above Green
  ( 1)  156.420/ 156.420 above Pairing
  ( 1)  15.859/ 15.859 above Renormalizing
  ( 1)   0.391/  0.391 above Substitution
```

In[ ]:=
```
BeginProfile[]
Timing[z2 = ρ2[Knot[12, NonAlternating, 369]]]
PrintProfile[]
```

Out[ ]=
```
ProfileRoot
```

Out[ ]=
$$\left\{467.438,\ \left\{-\frac{\left(1-T+T^2\right)\left(-1+T-2T^2+T^3\right)\left(-1+2T-T^2+T^3\right)}{T^4},\ -\frac{1}{T^8}(-1+T)^2\left(1-6T+20T^2-48T^3+82T^4-114T^5+134T^6-140T^7+134T^8-114T^9+82T^{10}-48T^{11}+20T^{12}-6T^{13}+T^{14}\right),\right.\right.$$
$$\left.\left.-\left(\left(1-16T+127T^2-668T^3+2631T^4-8324T^5+22282T^6-52780T^7+114992T^8-236376T^9+460598T^{10}-839688T^{11}+1404696T^{12}-2121524T^{13}+2862782T^{14}-3432312T^{15}+3647156T^{16}-3432312T^{17}+2862782T^{18}-2121524T^{19}+1404696T^{20}-839688T^{21}+460598T^{22}-236376T^{23}+114992T^{24}-52780T^{25}+22282T^{26}-8324T^{27}+2631T^{28}-668T^{29}+127T^{30}-16T^{31}+T^{32}\right)\Big/\left(2T^{12}\left(1-T+T^2\right)\left(-1+T-2T^2+T^3\right)\left(-1+2T-T^2+T^3\right)\right)\right)\right\}\right\}$$

Out[ ]=
```
ProfileRoot is root. Profiled time: 467.438
  ( 1)      0/ 467.440 above ρd
Pairing: called 1 times, time in 428.344/428.344
  ( 1)  428.340/ 428.340 under ρd
Renormalizing: called 1 times, time in 36.906/36.906
  ( 1)  36.906/ 36.906 under ρd
Substitution: called 1 times, time in 1.438/1.438
  ( 1)   1.438/  1.438 under ρd
Green: called 1 times, time in 0.75/0.75
  ( 1)   0.750/  0.750 under ρd
ρd: called 1 times, time in 0./467.438
  ( 1)      0/ 467.440 under ProfileRoot
  ( 1)   0.750/  0.750 above Green
  ( 1)  428.340/ 428.340 above Pairing
  ( 1)  36.906/ 36.906 above Renormalizing
  ( 1)   1.438/  1.438 above Substitution
```

In[ ]:=
```
Simplify[Thread[z1 == z2]]
```

Out[ ]=
$$\left\{\text{True},\ \text{True},\ \frac{(-1+T)\left(1-T+T^2\right)\left(1-3T+2T^2+5T^3-12T^4+18T^5-12T^6+5T^7+2T^8-3T^9+T^{10}\right)}{T}==0\right\}$$