

```
SetDirectory["C:\\drorbn\\AcademicPensive\\Talks\\Oaxaca-2210"];
```

```
In[*]:= Once[<< KnotTheory` ; << Rot.m];
```

Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.

Read more at <http://katlas.org/wiki/KnotTheory>.

Loading Rot.m from <http://drorbn.net/la22/ap> to compute rotation numbers.

```
In[*]:= V@γd,0[j_] := 0; V@γ1,φ[k_] := φ ( 1/2 - p̄k x̄k );
```

```
In[*]:= V@γ2,1[k_] := -1/2 p̄k x̄k; V@γ2,-1[k_] := -1/2 p̄k x̄k;
```

```
In[*]:= V@r1,s[i_, j_] :=
  s ( -1/2 + pi xi - pj xj + 1/2 (-1 + Ts) pi pj xi2 + 1/2 (1 - Ts) pj2 xi2 - pi pj xi xj + pj2 xi xj );
```

```
In[*]:= V@r2,1[i_, j_] := -1/2 pi xi + pj xi/2 + 1/4 (1 - 3 T) pi pj xi2 + 1/4 (-1 + 3 T) pj2 xi2 + 1/3 (-1 + T) pi2 pj xi3 -
  1/6 (-1 + T) (5 + T) pi pj2 xi3 + 1/6 (-1 + T) (3 + T) pj3 xi3 + 3/2 pi pj xi xj - 3/2 pj2 xi xj -
  1/2 pi2 pj xi2 xj + 1/2 (2 + T) pi pj2 xi2 xj + 1/2 (-1 - T) pj3 xi2 xj - 1/2 pi pj2 xi xj2 + 1/2 pj3 xi xj2;
```

```
In[*]:= V@r2,-1[i_, j_] := -1/2 pi xi + pj xi/2 + (-3 + T) pi pj xi2/4 T - (-3 + T) pj2 xi2/4 T - (-1 + T) pi2 pj xi3/3 T +
  (-1 + T) (1 + 5 T) pi pj2 xi3/6 T2 - (-1 + T) (1 + 3 T) pj3 xi3/6 T2 + 3/2 pi pj xi xj - 3/2 pj2 xi xj -
  1/2 pi2 pj xi2 xj + (1 + 2 T) pi pj2 xi2 xj/2 T - (1 + T) pj3 xi2 xj/2 T - 1/2 pi pj2 xi xj2 + 1/2 pj3 xi xj2;
```

```
In[*]:= {p*, x*, π*, ξ*, p̄*, x̄*, π̄*, ξ̄*} = {π, ε, p, x, π̄, ε̄, p̄, x̄}; (u-i-)* := (u*)i;
```

```
In[*]:= Zip{}[ε_] := ε;
```

```
Zip{ε, εs...}[ε_] := (Collect[ε // Zip{εs}, ε] /. f_ . εd . -> (D[f, {ε*, d}])) /. ε* -> 0
```

```
In[*]:= gPair[fs_, w_] := gPair[fs, w] = ZipJoin@@Table[{pα, p̄α, xα, x̄α}, {α, w}] [
  (Times@@(V/@fs)) Exp[Sum[gα,β (πα + π̄α) (εβ + ε̄β), {α, w}, {β, w}] - Sum[ε̄α πα, {α, w}]]]
```

```

In[ ]:=  $\rho_d[K_] := \text{Module}[\{Cs, \varphi, n, A, s, i, j, k, \Delta, G, d1, \rho d1, \rho d2, \rho d3\},$ 
  {Cs,  $\varphi$ } = Rot[K]; n = Length[Cs];
  A = IdentityMatrix[2 n + 1];
  Cases[Cs, {s_, i_, j_}  $\Rightarrow$  (A[[{i, j}, {i + 1, j + 1}]] +=  $\begin{pmatrix} -T^s & T^s - 1 \\ \theta & -1 \end{pmatrix}$ )]];
  G = Inverse[A];
   $\rho d1 = \text{Exp}[\text{Total}[\text{Cases}[Cs, \{s_, i_, j_} \Rightarrow \text{Sum}[e^{d1} r_{d1,s}[i, j], \{d1, d\}]]] +$ 
    Sum[ $e^{d1} \gamma_{d1,\varphi[[k]]}[k], \{k, 2 n\}, \{d1, d\}]]];$ 
   $\rho d2 = \text{Expand}[F[\{\}, \{\}] \times \text{Normal@Series}[\rho d1, \{e, \theta, d\}] // F[fs_, \{es\_ \}] \times$ 
    (f : (r |  $\gamma$ )ps_[is_])p_  $\Rightarrow$  F[Join[fs, Table[f, p]], DeleteDuplicates@{es, is}]];
   $\rho d3 = \rho d2 / F[fs_, es_] \Rightarrow$  (gPair[
    Replace[fs, Thread[es  $\rightarrow$  Range@Length@es], {2}], Length@es
  ] /  $\cdot g_{\alpha, \beta} \Rightarrow g_{es[[\alpha]], es[[\beta]]}$ );
  Collect[
    { $\Delta, \rho d3 / e \rightarrow e \Delta^2$ } /  $\cdot \{g_{\alpha, \beta} \Rightarrow G[[\alpha, \beta]], \Delta \rightarrow T^{(-\text{Total}[\varphi] - \text{Total}[Cs[[All, 1]])] / 2} \text{Det}[A]\},$ 
    e, Factor]
  ];

```

```

In[ ]:= {K = Knot[3, 1], d = 2}

```

```

Out[ ]:= {Knot[3, 1], 2}

```

```
In[*]:= {Cs, φ} = Rot[K]; n = Length[Cs];
A = IdentityMatrix[2 n + 1];
Cases[Cs, {s_, i_, j_} => (A[[{i, j}, {i + 1, j + 1}]] += (
  -T^s T^s - 1
  0 -1
))]];
G = Inverse[A]
```

KnotTheory: Loading precomputed data in PD4Knots`.

Out[*]=

$$\left\{ \left\{ 1, \frac{1 + \frac{1}{T^2} - \frac{1}{T}}{\frac{1}{T^3} - \frac{1}{T^2} + \frac{1}{T}}, 1, \frac{1 + \frac{1}{T^2} - \frac{1}{T}}{\frac{1}{T^3} - \frac{1}{T^2} + \frac{1}{T}}, 1, \frac{1 + \frac{1}{T^2} - \frac{1}{T}}{\frac{1}{T^3} - \frac{1}{T^2} + \frac{1}{T}}, 1 \right\}, \right.$$

$$\left. \left\{ 0, \frac{1 + \frac{1}{T^2} - \frac{1}{T}}{\frac{1}{T^3} - \frac{1}{T^2} + \frac{1}{T}}, 1, \frac{1 + \frac{1}{T^2} - \frac{1}{T}}{\frac{1}{T^3} - \frac{1}{T^2} + \frac{1}{T}}, 1, \frac{1 + \frac{1}{T^2} - \frac{1}{T}}{\frac{1}{T^3} - \frac{1}{T^2} + \frac{1}{T}}, 1 \right\}, \right.$$

$$\left. \left\{ 0, \frac{1 - \frac{1}{T}}{\frac{1}{T^3} - \frac{1}{T^2} + \frac{1}{T}}, \frac{\frac{1}{T^3} - \frac{2}{T^2} + \frac{2}{T}}{\frac{1}{T^3} - \frac{1}{T^2} + \frac{1}{T}}, \frac{1}{\frac{1}{T^3} - \frac{1}{T^2} + \frac{1}{T}}, \frac{1}{\left(\frac{1}{T^3} - \frac{1}{T^2} + \frac{1}{T}\right) T}, \frac{1 + \frac{1}{T^2} - \frac{1}{T}}{\frac{1}{T^3} - \frac{1}{T^2} + \frac{1}{T}}, 1 \right\}, \right.$$

$$\left. \left\{ 0, \frac{1 - \frac{1}{T}}{\frac{1}{T^3} - \frac{1}{T^2} + \frac{1}{T}}, \frac{-\frac{1}{T^2} + \frac{1}{T}}{\frac{1}{T^3} - \frac{1}{T^2} + \frac{1}{T}}, \frac{1}{\frac{1}{T^3} - \frac{1}{T^2} + \frac{1}{T}}, \frac{1}{\left(\frac{1}{T^3} - \frac{1}{T^2} + \frac{1}{T}\right) T}, \frac{1 + \frac{1}{T^2} - \frac{1}{T}}{\frac{1}{T^3} - \frac{1}{T^2} + \frac{1}{T}}, 1 \right\}, \right.$$

$$\left. \left\{ 0, \frac{1 + \frac{1}{T^2} - \frac{2}{T}}{\frac{1}{T^3} - \frac{1}{T^2} + \frac{1}{T}}, \frac{\frac{1}{T^3} - \frac{2}{T^2} + \frac{1}{T}}{\frac{1}{T^3} - \frac{1}{T^2} + \frac{1}{T}}, \frac{1 - \frac{1}{T}}{\frac{1}{T^3} - \frac{1}{T^2} + \frac{1}{T}}, \frac{\frac{1}{T^3} - \frac{2}{T^2} + \frac{2}{T}}{\frac{1}{T^3} - \frac{1}{T^2} + \frac{1}{T}}, \frac{1 + \frac{1}{T^2} - \frac{1}{T}}{\frac{1}{T^3} - \frac{1}{T^2} + \frac{1}{T}}, 1 \right\}, \right.$$

$$\left. \left\{ 0, \frac{1 + \frac{1}{T^2} - \frac{2}{T}}{\frac{1}{T^3} - \frac{1}{T^2} + \frac{1}{T}}, \frac{\frac{1}{T^3} - \frac{2}{T^2} + \frac{1}{T}}{\frac{1}{T^3} - \frac{1}{T^2} + \frac{1}{T}}, \frac{1 - \frac{1}{T}}{\frac{1}{T^3} - \frac{1}{T^2} + \frac{1}{T}}, \frac{-\frac{1}{T^2} + \frac{1}{T}}{\frac{1}{T^3} - \frac{1}{T^2} + \frac{1}{T}}, \frac{1 + \frac{1}{T^2} - \frac{1}{T}}{\frac{1}{T^3} - \frac{1}{T^2} + \frac{1}{T}}, 1 \right\}, \{0, 0, 0, 0, 0, 0, 1\} \right\}$$

```
In[*]:= ρd1 = Exp[Total[Cases[Cs, {s_, i_, j_} => Sum[e^d1 r_{d1,s}[i, j], {d1, d}]]] +
Sum[e^d1 γ_{d1,φ[[k]]}[k], {k, 2 n}, {d1, d}]]]
```

Out[*]=

$$e^{\epsilon} r_{1,-1}[2,5] + \epsilon r_{1,-1}[4,1] + \epsilon r_{1,-1}[6,3] + \epsilon^2 r_{2,-1}[2,5] + \epsilon^2 r_{2,-1}[4,1] + \epsilon^2 r_{2,-1}[6,3] + \epsilon \gamma_{1,-1}[4] + \epsilon \gamma_{1,0}[1] + \epsilon \gamma_{1,0}[2] + \epsilon \gamma_{1,0}[3] + \epsilon \gamma_{1,0}[5] + \epsilon \gamma_{1,0}[\epsilon]$$

In[*]:= `pd2 = Expand[Normal@Series[ρd1, {ε, 0, d}]]`

Out[*]=

$$\begin{aligned}
 & 1 + \epsilon r_{1,-1}[2, 5] + \frac{1}{2} \epsilon^2 r_{1,-1}[2, 5]^2 + \epsilon r_{1,-1}[4, 1] + \epsilon^2 r_{1,-1}[2, 5] r_{1,-1}[4, 1] + \frac{1}{2} \epsilon^2 r_{1,-1}[4, 1]^2 + \\
 & \epsilon r_{1,-1}[6, 3] + \epsilon^2 r_{1,-1}[2, 5] r_{1,-1}[6, 3] + \epsilon^2 r_{1,-1}[4, 1] r_{1,-1}[6, 3] + \frac{1}{2} \epsilon^2 r_{1,-1}[6, 3]^2 + \\
 & \epsilon^2 r_{2,-1}[2, 5] + \epsilon^2 r_{2,-1}[4, 1] + \epsilon^2 r_{2,-1}[6, 3] + \epsilon \gamma_{1,-1}[4] + \epsilon^2 r_{1,-1}[2, 5] \gamma_{1,-1}[4] + \\
 & \epsilon^2 r_{1,-1}[4, 1] \gamma_{1,-1}[4] + \epsilon^2 r_{1,-1}[6, 3] \gamma_{1,-1}[4] + \frac{1}{2} \epsilon^2 \gamma_{1,-1}[4]^2 + \epsilon \gamma_{1,0}[1] + \epsilon^2 r_{1,-1}[2, 5] \gamma_{1,0}[1] + \\
 & \epsilon^2 r_{1,-1}[4, 1] \gamma_{1,0}[1] + \epsilon^2 r_{1,-1}[6, 3] \gamma_{1,0}[1] + \epsilon^2 \gamma_{1,-1}[4] \gamma_{1,0}[1] + \frac{1}{2} \epsilon^2 \gamma_{1,0}[1]^2 + \epsilon \gamma_{1,0}[2] + \\
 & \epsilon^2 r_{1,-1}[2, 5] \gamma_{1,0}[2] + \epsilon^2 r_{1,-1}[4, 1] \gamma_{1,0}[2] + \epsilon^2 r_{1,-1}[6, 3] \gamma_{1,0}[2] + \epsilon^2 \gamma_{1,-1}[4] \gamma_{1,0}[2] + \\
 & \epsilon^2 \gamma_{1,0}[1] \gamma_{1,0}[2] + \frac{1}{2} \epsilon^2 \gamma_{1,0}[2]^2 + \epsilon \gamma_{1,0}[3] + \epsilon^2 r_{1,-1}[2, 5] \gamma_{1,0}[3] + \epsilon^2 r_{1,-1}[4, 1] \gamma_{1,0}[3] + \\
 & \epsilon^2 r_{1,-1}[6, 3] \gamma_{1,0}[3] + \epsilon^2 \gamma_{1,-1}[4] \gamma_{1,0}[3] + \epsilon^2 \gamma_{1,0}[1] \gamma_{1,0}[3] + \epsilon^2 \gamma_{1,0}[2] \gamma_{1,0}[3] + \\
 & \frac{1}{2} \epsilon^2 \gamma_{1,0}[3]^2 + \epsilon \gamma_{1,0}[5] + \epsilon^2 r_{1,-1}[2, 5] \gamma_{1,0}[5] + \epsilon^2 r_{1,-1}[4, 1] \gamma_{1,0}[5] + \epsilon^2 r_{1,-1}[6, 3] \gamma_{1,0}[5] + \\
 & \epsilon^2 \gamma_{1,-1}[4] \gamma_{1,0}[5] + \epsilon^2 \gamma_{1,0}[1] \gamma_{1,0}[5] + \epsilon^2 \gamma_{1,0}[2] \gamma_{1,0}[5] + \epsilon^2 \gamma_{1,0}[3] \gamma_{1,0}[5] + \frac{1}{2} \epsilon^2 \gamma_{1,0}[5]^2 + \\
 & \epsilon \gamma_{1,0}[6] + \epsilon^2 r_{1,-1}[2, 5] \gamma_{1,0}[6] + \epsilon^2 r_{1,-1}[4, 1] \gamma_{1,0}[6] + \epsilon^2 r_{1,-1}[6, 3] \gamma_{1,0}[6] + \\
 & \epsilon^2 \gamma_{1,-1}[4] \gamma_{1,0}[6] + \epsilon^2 \gamma_{1,0}[1] \gamma_{1,0}[6] + \epsilon^2 \gamma_{1,0}[2] \gamma_{1,0}[6] + \epsilon^2 \gamma_{1,0}[3] \gamma_{1,0}[6] + \epsilon^2 \gamma_{1,0}[5] \gamma_{1,0}[6] + \\
 & \frac{1}{2} \epsilon^2 \gamma_{1,0}[6]^2 + \epsilon^2 \gamma_{2,-1}[4] + \epsilon^2 \gamma_{2,0}[1] + \epsilon^2 \gamma_{2,0}[2] + \epsilon^2 \gamma_{2,0}[3] + \epsilon^2 \gamma_{2,0}[5] + \epsilon^2 \gamma_{2,0}[6]
 \end{aligned}$$

In[*]:= `pd2 = Expand[F[{} , {}] × Normal@Series[ρd1, {ε, 0, d}]]`

Out[*]=

$$\begin{aligned}
 & F[{} , {}] + \epsilon F[{} , {}] r_{1,-1}[2, 5] + \frac{1}{2} \epsilon^2 F[{} , {}] r_{1,-1}[2, 5]^2 + \epsilon F[{} , {}] r_{1,-1}[4, 1] + \\
 & \epsilon^2 F[{} , {}] r_{1,-1}[2, 5] r_{1,-1}[4, 1] + \frac{1}{2} \epsilon^2 F[{} , {}] r_{1,-1}[4, 1]^2 + \epsilon F[{} , {}] r_{1,-1}[6, 3] + \\
 & \epsilon^2 F[{} , {}] r_{1,-1}[2, 5] r_{1,-1}[6, 3] + \epsilon^2 F[{} , {}] r_{1,-1}[4, 1] r_{1,-1}[6, 3] + \\
 & \frac{1}{2} \epsilon^2 F[{} , {}] r_{1,-1}[6, 3]^2 + \epsilon^2 F[{} , {}] r_{2,-1}[2, 5] + \epsilon^2 F[{} , {}] r_{2,-1}[4, 1] + \\
 & \epsilon^2 F[{} , {}] r_{2,-1}[6, 3] + \epsilon F[{} , {}] \gamma_{1,-1}[4] + \epsilon^2 F[{} , {}] r_{1,-1}[2, 5] \gamma_{1,-1}[4] + \\
 & \epsilon^2 F[{} , {}] r_{1,-1}[4, 1] \gamma_{1,-1}[4] + \epsilon^2 F[{} , {}] r_{1,-1}[6, 3] \gamma_{1,-1}[4] + \frac{1}{2} \epsilon^2 F[{} , {}] \gamma_{1,-1}[4]^2 + \\
 & \epsilon F[{} , {}] \gamma_{1,0}[1] + \epsilon^2 F[{} , {}] r_{1,-1}[2, 5] \gamma_{1,0}[1] + \epsilon^2 F[{} , {}] r_{1,-1}[4, 1] \gamma_{1,0}[1] + \\
 & \epsilon^2 F[{} , {}] r_{1,-1}[6, 3] \gamma_{1,0}[1] + \epsilon^2 F[{} , {}] \gamma_{1,-1}[4] \gamma_{1,0}[1] + \frac{1}{2} \epsilon^2 F[{} , {}] \gamma_{1,0}[1]^2 + \\
 & \epsilon F[{} , {}] \gamma_{1,0}[2] + \epsilon^2 F[{} , {}] r_{1,-1}[2, 5] \gamma_{1,0}[2] + \epsilon^2 F[{} , {}] r_{1,-1}[4, 1] \gamma_{1,0}[2] + \\
 & \epsilon^2 F[{} , {}] r_{1,-1}[6, 3] \gamma_{1,0}[2] + \epsilon^2 F[{} , {}] \gamma_{1,-1}[4] \gamma_{1,0}[2] + \epsilon^2 F[{} , {}] \gamma_{1,0}[1] \gamma_{1,0}[2] + \\
 & \frac{1}{2} \epsilon^2 F[{} , {}] \gamma_{1,0}[2]^2 + \epsilon F[{} , {}] \gamma_{1,0}[3] + \epsilon^2 F[{} , {}] r_{1,-1}[2, 5] \gamma_{1,0}[3] + \\
 & \epsilon^2 F[{} , {}] r_{1,-1}[4, 1] \gamma_{1,0}[3] + \epsilon^2 F[{} , {}] r_{1,-1}[6, 3] \gamma_{1,0}[3] + \epsilon^2 F[{} , {}] \gamma_{1,-1}[4] \gamma_{1,0}[3] + \\
 & \epsilon^2 F[{} , {}] \gamma_{1,0}[1] \gamma_{1,0}[3] + \epsilon^2 F[{} , {}] \gamma_{1,0}[2] \gamma_{1,0}[3] + \frac{1}{2} \epsilon^2 F[{} , {}] \gamma_{1,0}[3]^2 + \\
 & \epsilon F[{} , {}] \gamma_{1,0}[5] + \epsilon^2 F[{} , {}] r_{1,-1}[2, 5] \gamma_{1,0}[5] + \epsilon^2 F[{} , {}] r_{1,-1}[4, 1] \gamma_{1,0}[5] + \\
 & \epsilon^2 F[{} , {}] r_{1,-1}[6, 3] \gamma_{1,0}[5] + \epsilon^2 F[{} , {}] \gamma_{1,-1}[4] \gamma_{1,0}[5] + \epsilon^2 F[{} , {}] \gamma_{1,0}[1] \gamma_{1,0}[5] + \\
 & \epsilon^2 F[{} , {}] \gamma_{1,0}[2] \gamma_{1,0}[5] + \epsilon^2 F[{} , {}] \gamma_{1,0}[3] \gamma_{1,0}[5] + \frac{1}{2} \epsilon^2 F[{} , {}] \gamma_{1,0}[5]^2 + \\
 & \epsilon F[{} , {}] \gamma_{1,0}[6] + \epsilon^2 F[{} , {}] r_{1,-1}[2, 5] \gamma_{1,0}[6] + \epsilon^2 F[{} , {}] r_{1,-1}[4, 1] \gamma_{1,0}[6] + \\
 & \epsilon^2 F[{} , {}] r_{1,-1}[6, 3] \gamma_{1,0}[6] + \epsilon^2 F[{} , {}] \gamma_{1,-1}[4] \gamma_{1,0}[6] + \epsilon^2 F[{} , {}] \gamma_{1,0}[1] \gamma_{1,0}[6] + \\
 & \epsilon^2 F[{} , {}] \gamma_{1,0}[2] \gamma_{1,0}[6] + \epsilon^2 F[{} , {}] \gamma_{1,0}[3] \gamma_{1,0}[6] + \epsilon^2 F[{} , {}] \gamma_{1,0}[5] \gamma_{1,0}[6] + \\
 & \frac{1}{2} \epsilon^2 F[{} , {}] \gamma_{1,0}[6]^2 + \epsilon^2 F[{} , {}] \gamma_{2,-1}[4] + \epsilon^2 F[{} , {}] \gamma_{2,0}[1] + \\
 & \epsilon^2 F[{} , {}] \gamma_{2,0}[2] + \epsilon^2 F[{} , {}] \gamma_{2,0}[3] + \epsilon^2 F[{} , {}] \gamma_{2,0}[5] + \epsilon^2 F[{} , {}] \gamma_{2,0}[6]
 \end{aligned}$$

In[*]:= `pd2 = Expand[F[{}], {}] × Normal@Series[pd1, {ε, 0, d}] //.`

`F[fs_, {es_}] × (f : (r | γ)ps_[is_])p_ ⇒
F[Join[fs, Table[f, p]], DeleteDuplicates@{es, is}]`

Out[*]=

$$\begin{aligned}
 & F[{}], {} + \epsilon F[\{r_{1,-1}[2, 5]\}, \{2, 5\}] + \epsilon F[\{r_{1,-1}[4, 1]\}, \{4, 1\}] + \\
 & \epsilon F[\{r_{1,-1}[6, 3]\}, \{6, 3\}] + \epsilon^2 F[\{r_{2,-1}[2, 5]\}, \{2, 5\}] + \epsilon^2 F[\{r_{2,-1}[4, 1]\}, \{4, 1\}] + \\
 & \epsilon^2 F[\{r_{2,-1}[6, 3]\}, \{6, 3\}] + \epsilon F[\{\gamma_{1,-1}[4]\}, \{4\}] + \epsilon F[\{\gamma_{1,0}[1]\}, \{1\}] + \\
 & \epsilon F[\{\gamma_{1,0}[2]\}, \{2\}] + \epsilon F[\{\gamma_{1,0}[3]\}, \{3\}] + \epsilon F[\{\gamma_{1,0}[5]\}, \{5\}] + \epsilon F[\{\gamma_{1,0}[6]\}, \{6\}] + \\
 & \epsilon^2 F[\{\gamma_{2,-1}[4]\}, \{4\}] + \epsilon^2 F[\{\gamma_{2,0}[1]\}, \{1\}] + \epsilon^2 F[\{\gamma_{2,0}[2]\}, \{2\}] + \epsilon^2 F[\{\gamma_{2,0}[3]\}, \{3\}] + \\
 & \epsilon^2 F[\{\gamma_{2,0}[5]\}, \{5\}] + \epsilon^2 F[\{\gamma_{2,0}[6]\}, \{6\}] + \frac{1}{2} \epsilon^2 F[\{r_{1,-1}[2, 5], r_{1,-1}[2, 5]\}, \{2, 5\}] + \\
 & \epsilon^2 F[\{r_{1,-1}[2, 5], r_{1,-1}[4, 1]\}, \{2, 5, 4, 1\}] + \epsilon^2 F[\{r_{1,-1}[2, 5], r_{1,-1}[6, 3]\}, \{2, 5, 6, 3\}] + \\
 & \epsilon^2 F[\{r_{1,-1}[2, 5], \gamma_{1,-1}[4]\}, \{2, 5, 4\}] + \epsilon^2 F[\{r_{1,-1}[2, 5], \gamma_{1,0}[1]\}, \{2, 5, 1\}] + \\
 & \epsilon^2 F[\{r_{1,-1}[2, 5], \gamma_{1,0}[2]\}, \{2, 5\}] + \epsilon^2 F[\{r_{1,-1}[2, 5], \gamma_{1,0}[3]\}, \{2, 5, 3\}] + \\
 & \epsilon^2 F[\{r_{1,-1}[2, 5], \gamma_{1,0}[5]\}, \{2, 5\}] + \epsilon^2 F[\{r_{1,-1}[2, 5], \gamma_{1,0}[6]\}, \{2, 5, 6\}] + \\
 & \frac{1}{2} \epsilon^2 F[\{r_{1,-1}[4, 1], r_{1,-1}[4, 1]\}, \{4, 1\}] + \epsilon^2 F[\{r_{1,-1}[4, 1], r_{1,-1}[6, 3]\}, \{4, 1, 6, 3\}] + \\
 & \epsilon^2 F[\{r_{1,-1}[4, 1], \gamma_{1,-1}[4]\}, \{4, 1\}] + \epsilon^2 F[\{r_{1,-1}[4, 1], \gamma_{1,0}[1]\}, \{4, 1\}] + \\
 & \epsilon^2 F[\{r_{1,-1}[4, 1], \gamma_{1,0}[2]\}, \{4, 1, 2\}] + \epsilon^2 F[\{r_{1,-1}[4, 1], \gamma_{1,0}[3]\}, \{4, 1, 3\}] + \\
 & \epsilon^2 F[\{r_{1,-1}[4, 1], \gamma_{1,0}[5]\}, \{4, 1, 5\}] + \epsilon^2 F[\{r_{1,-1}[4, 1], \gamma_{1,0}[6]\}, \{4, 1, 6\}] + \\
 & \frac{1}{2} \epsilon^2 F[\{r_{1,-1}[6, 3], r_{1,-1}[6, 3]\}, \{6, 3\}] + \epsilon^2 F[\{r_{1,-1}[6, 3], \gamma_{1,-1}[4]\}, \{6, 3, 4\}] + \\
 & \epsilon^2 F[\{r_{1,-1}[6, 3], \gamma_{1,0}[1]\}, \{6, 3, 1\}] + \epsilon^2 F[\{r_{1,-1}[6, 3], \gamma_{1,0}[2]\}, \{6, 3, 2\}] + \\
 & \epsilon^2 F[\{r_{1,-1}[6, 3], \gamma_{1,0}[3]\}, \{6, 3\}] + \epsilon^2 F[\{r_{1,-1}[6, 3], \gamma_{1,0}[5]\}, \{6, 3, 5\}] + \\
 & \epsilon^2 F[\{r_{1,-1}[6, 3], \gamma_{1,0}[6]\}, \{6, 3\}] + \frac{1}{2} \epsilon^2 F[\{\gamma_{1,-1}[4], \gamma_{1,-1}[4]\}, \{4\}] + \\
 & \epsilon^2 F[\{\gamma_{1,-1}[4], \gamma_{1,0}[1]\}, \{4, 1\}] + \epsilon^2 F[\{\gamma_{1,-1}[4], \gamma_{1,0}[2]\}, \{4, 2\}] + \\
 & \epsilon^2 F[\{\gamma_{1,-1}[4], \gamma_{1,0}[3]\}, \{4, 3\}] + \epsilon^2 F[\{\gamma_{1,-1}[4], \gamma_{1,0}[5]\}, \{4, 5\}] + \\
 & \epsilon^2 F[\{\gamma_{1,-1}[4], \gamma_{1,0}[6]\}, \{4, 6\}] + \frac{1}{2} \epsilon^2 F[\{\gamma_{1,0}[1], \gamma_{1,0}[1]\}, \{1\}] + \\
 & \epsilon^2 F[\{\gamma_{1,0}[1], \gamma_{1,0}[2]\}, \{1, 2\}] + \epsilon^2 F[\{\gamma_{1,0}[1], \gamma_{1,0}[3]\}, \{1, 3\}] + \\
 & \epsilon^2 F[\{\gamma_{1,0}[1], \gamma_{1,0}[5]\}, \{1, 5\}] + \epsilon^2 F[\{\gamma_{1,0}[1], \gamma_{1,0}[6]\}, \{1, 6\}] + \\
 & \frac{1}{2} \epsilon^2 F[\{\gamma_{1,0}[2], \gamma_{1,0}[2]\}, \{2\}] + \epsilon^2 F[\{\gamma_{1,0}[2], \gamma_{1,0}[3]\}, \{2, 3\}] + \\
 & \epsilon^2 F[\{\gamma_{1,0}[2], \gamma_{1,0}[5]\}, \{2, 5\}] + \epsilon^2 F[\{\gamma_{1,0}[2], \gamma_{1,0}[6]\}, \{2, 6\}] + \\
 & \frac{1}{2} \epsilon^2 F[\{\gamma_{1,0}[3], \gamma_{1,0}[3]\}, \{3\}] + \epsilon^2 F[\{\gamma_{1,0}[3], \gamma_{1,0}[5]\}, \{3, 5\}] + \\
 & \epsilon^2 F[\{\gamma_{1,0}[3], \gamma_{1,0}[6]\}, \{3, 6\}] + \frac{1}{2} \epsilon^2 F[\{\gamma_{1,0}[5], \gamma_{1,0}[5]\}, \{5\}] + \\
 & \epsilon^2 F[\{\gamma_{1,0}[5], \gamma_{1,0}[6]\}, \{5, 6\}] + \frac{1}{2} \epsilon^2 F[\{\gamma_{1,0}[6], \gamma_{1,0}[6]\}, \{6\}]
 \end{aligned}$$

```
In[*]:= gPair[fs_, w_] := gPair[fs, w] = ZipJoin@@Table[{pα, p̄α, xα, x̄α}, {α, w}] [
  (Times@@(V/@fs)) Exp[Sum[gα,β (πα + π̄α) (ξβ + ξ̄β), {α, w}, {β, w}] - Sum[ξ̄α πα, {α, w}]]]
```

```
In[*]:= ρd3 = ρd2 /. F[fs_, es_] => (gPair[
  Replace[fs, Thread[es -> Range@Length@es], {2}], Length@es
] /. gα,β -> ges[[α]], es[[β]])
```

Out[*]=

$$1 + \epsilon \left(-\frac{1}{2} + g_{4,4} \right) - \frac{1}{2} \epsilon^2 g_4 \dots + \dots + 14 \dots +$$

$$\epsilon^2 \left(\dots + \frac{1}{T} \dots \right) + \frac{1}{2} \epsilon^2 \left(\frac{1}{4} + g_{3,6} - 2 g_{3,3} g_{3,6} + \dots + 37 \dots \right) +$$

$$4 g_{3,3}^2 g_{6,6}^2 + 6 \left(-1 + \frac{1}{T} \right) g_{3,6} g_{6,6}^2 + 12 \left(1 - \frac{1}{T} \right) g_{3,3} g_{3,6} g_{6,6}^2 + 6 \left(-1 + \frac{1}{T} \right)^2 g_{3,6}^2 g_{6,6}^2$$

Full expression not available (original memory size: 0.4 MB)

```
In[*]:= ?gPair
```

```
In[*]:= Collect[
  {Δ, ρd3 /. ε -> ε Δ^2} /. {gα,β -> G[[α, β]], Δ -> T^(-Total[φ] - Total[Cs[[All, 1]]]) / 2 Det[A]},
  ε, Factor]
```

Out[*]=

$$\left\{ \frac{1 - T + T^2}{T}, 1 + \frac{(-1 + T)^2 (1 + T^2) \epsilon}{T^2} + \frac{(1 - 4T + 7T^2 - 12T^3 + 18T^4 - 12T^5 + 7T^6 - 4T^7 + T^8) \epsilon^2}{2T^4} \right\}$$