Pensieve header: Finding the Gauss-Gassner-Alexander formula. Seeded at pensieve://2015-07/PolyPoly/.

```
SetDirectory["C:/drorbn/AcademicPensieve/Talks/NCSU-1604/"];
<< KnotTheory`
```

```
Loading KnotTheory` version of September 6, 2014, 13:37:37.2841.
Read more at http://katlas.org/wiki/KnotTheory.
```

```
Xs[xs_Xs] := xs;
Xs[L_] :=
  Xs @@ PD[L] /. X[i_, j_, k_, l_] :> If[PositiveQ[X[i, j, k, l]], Xp[l, i], Xm[j, i]];
```

Initialization

```
ΓCollect[Γ[ω_, λ_]] := Γ[Simplify[ω],
   Collect[λ, h_, Collect[#, t_, Factor] &]];
Format[Γ[ω_, λ_]] := Module[{S, M},
   S = Union@Cases[Γ[ω, λ], (h | t)_a_ :> a, ∞];
   M = Outer[Factor[∂_{h_#1 t_#2} λ] &, S, S];
   M = Prepend[M, t_# & /@ S] // Transpose;
   M = Prepend[M, Prepend[h_# & /@ S, ω]];
   M // MatrixForm];
```

Program

```
Γ /: Γ[ω1_, λ1_] Γ[ω2_, λ2_] := Γ[ω1 * ω2, λ1 + λ2];
m_{a_ b_→c_}[Γ[ω_, λ_]] := Module[{α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ},
   ( α β θ            ( ∂_{t_a,h_a} λ   ∂_{t_a,h_b} λ   ∂_{t_a} λ
     γ δ ϵ      =       ∂_{t_b,h_a} λ   ∂_{t_b,h_b} λ   ∂_{t_b} λ
     ϕ ψ Ξ )            ∂_{h_a} λ       ∂_{h_b} λ       λ        ) /. (t | h)_{a|b} → 0;
   Γ[(μ = 1 - β) ω, {t_c, 1}.( γ + α δ / μ   ϵ + δ θ / μ
                                ϕ + α ψ / μ   Ξ + ψ θ / μ ).{h_c, 1}]
     /. {T_a → T_c, T_b → T_c} // ΓCollect];
Rp_{a_ b_} := Γ[1, Tr[(t_a, t_b)^T.(1  1 - T_a
                                    0  T_a   ).(h_a, h_b)]];
Rm_{a_ b_} := Rp_{ab} /. T_a → 1/T_a;
ϵ_{a_} := Γ[1, t_a h_a];
```

MetaAssoc

```
ζ = Γ[ω, Tr[(t1, t2, t3, tS)^T.( α11 α12 α13 θ1
                                  α21 α22 α23 θ2
                                  α31 α32 α33 θ3
                                  ϕ1  ϕ2  ϕ3  Ξ  ).(h1, h2, h3, hS)]];
(ζ // m_{12→1} // m_{13→1}) == (ζ // m_{23→2} // m_{12→1})
```

MetaAssoc

```
True
```

R3

```
{Rm51 Rm62 Rp34 // m14→1 // m25→2 // m36→3,
  Rp61 Rm24 Rm35 // m14→1 // m25→2 // m36→3}
```

R3

$$\left\{\begin{pmatrix} 1 & h_1 & h_2 & h_3 \\ t_1 & \frac{T_3}{T_2} & 0 & 0 \\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0 \\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1 \end{pmatrix}, \begin{pmatrix} 1 & h_1 & h_2 & h_3 \\ t_1 & \frac{T_3}{T_2} & 0 & 0 \\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0 \\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1 \end{pmatrix}\right\}$$

8_17

```
z = Rm12,1 Rm27 Rm83 Rm4,11 Rp16,5 Rp6,13 Rp14,9 Rp10,15;
Do[z = z // m1k→1, {k, 2, 16}];
z
```

8_17

$$\begin{pmatrix} 11 - \frac{1}{T_1^3} + \frac{4}{T_1^2} - \frac{8}{T_1} - 8\,T_1 + 4\,T_1^2 - T_1^3 & h_1 \\ t_1 & 1 \end{pmatrix}$$

Extras

Four types of R1

```
{Rp12 // m12→1, Rp12 // m21→1, Rm12 // m12→1, Rm12 // m21→1}
```

$$\left\{\begin{pmatrix} T_1 & h_1 \\ t_1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & h_1 \\ t_1 & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{T_1} & h_1 \\ t_1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & h_1 \\ t_1 & 1 \end{pmatrix}\right\}$$

Two types of R2

```
{Rp12 Rm34 // m13→1 // m24→2, Rp12 Rm34 // m13→1 // m42→2}
```

$$\left\{\begin{pmatrix} 1 & h_1 & h_2 \\ t_1 & 1 & 0 \\ t_2 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & h_1 & h_2 \\ t_1 & 1 & 0 \\ t_2 & 0 & 1 \end{pmatrix}\right\}$$

Main

```
Xs[K = Knot[8, 17]]
```

KnotTheory::loading : Loading precomputed data in PD4Knots`.

```
Xs[Xp[1, 6], Xp[7, 14], Xm[3, 8], Xm[13, 2], Xm[5, 12], Xm[9, 4], Xp[11, 16], Xp[15, 10]]
```

```
GG[xs_Xs, k_, F_] := Module[{xl, len, y, cuts, pcuts, γ, λ},
  xl = List @@ xs; len = 2 Length@xs;
  Sum[
   cuts = Union @@ (List @@@ y);
   F[
    y /. Thread[cuts → Range[Length@cuts]],
    γ = ϵ_{len+1} Times @@ xl /. {Xp[a_, b_] :→ Rp_{ab}, Xm[a_, b_] :→ Rm_{ab}} /. T_ → T;
    Do[
     If[! MemberQ[cuts, j], γ = γ // m_{j,j+1→j+1}],
     {j, len}
    ];
    λ = γ[[2]];
    Table[Simplify[∂_{t_a,h_b} λ], {a, cuts ⋃ {len + 1}}, {b, cuts ⋃ {len + 1}}]
   ],
   {y, Subsets[xl, k]}
  ]
 ];
GG[K_, k_, F_] := GG[Xs[K], k, F];
```

```
GG[K, {1}, F] // Short
```

$$F\left[\{Xm[1,2]\},\left\{\left\{-\frac{\left(1-T+T^2\right)^2\left(-1+T-2T^2+T^3\right)}{T^4},\frac{(-1+T)\left(-1+\ll 9\gg+T^7\right)}{T^4},-\frac{(-1+T)^2\left(1-T+3T^2-T^3+T^4\right)}{T^3}\right\},\{\ll 1\gg\},\left\{\frac{(-1+T)\left(\ll 1\gg\ \ll 1\gg^2\right)}{T^2},-\frac{\ll 1\gg}{\ll 1\gg},-3+\frac{1}{T}+4T-2T^2+T^3\right\}\right\}\right]+\ll 6\gg+\ll 1\gg$$

```
F1[xs_List, m_] := F[xs, MatrixForm[m /. T → 1]];
GG[K, {2}, F1] // Short
```

$$F\left[\{Xm[1,2],Xp[3,4]\},\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}\right]+F\left[\{Xm[1,2],Xp[4,3]\},\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}\right]+\ll 19\gg+$$
$$2F\left[\{Xp[2,4],Xm[3,1]\},\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}\right]+F\left[\{Xp[2,4],Xp[3,1]\},\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}\right]$$

```
F0[xs_List, m_] := F[xs, MatrixForm[m]];
GG[K, {1}, F0] // Short
```

```
F[{Xm[1, 2]}, (≪1≫)] + F[{Xm[1, 2]}, (≪1≫)] +
 F[{Xm[2, 1]}, (≪1≫)] + F[{Xm[2, 1]}, (≪1≫)] + ≪1≫ +
 F[{Xp[1, 2]}, (≪1≫)] + F[{Xp[1, 2]}, (≪1≫)] + F[{Xp[2, 1]}, (≪1≫)]
```

```
FA[{x_}, m_] := Module[{a11, a12, a13, a21, a22, a23, a31, a32, a33},
  (a11 a12 a13
   a21 a22 a23
   a31 a32 a33) = m;
  Simplify[Times[
    If[Head[x] === Xp, +1, -1],
    If[x[[1]] == 1, (-a23 a32 + a22 a33)/(a13 a32 + a33 - a12 a33), (a13 a32 - a12 a33)/(a32 - a23 a32 + a22 a33)]
   ]]
 ]
```

```
GG[K, {1}, FA] // Simplify
```

$$\frac{-2+4\,T-11\,T^3+16\,T^4-12\,T^5+4\,T^6}{1-4\,T+8\,T^2-11\,T^3+8\,T^4-4\,T^5+T^6}$$

```
Alexander[K][T]
```

$$11-\frac{1}{T^3}+\frac{4}{T^2}-\frac{8}{T}-8\,T+4\,T^2-T^3$$

```
Table[
  ZK = Times @@ Xs[K] /. {Xp[a_, b_] :> Rp_ab, Xm[a_, b_] :> Rm_ab} /. T_ → T;
  Do[ZK = ZK // m_{1,k→1}, {k, 2, 2 Length[Xs[K]]}];
  ZK = ZK[[1]];
  alex = Alexander[K][T];
  gg = GG[K, {1}, FA];
  {K, ZK, T * D[Log[ZK], T], gg , alex, gg - T ∂_T Log[alex]} // Factor // Simplify,
  {K, AllKnots[{3, 8}]}
 ] // MatrixForm
```

Knot[3, 1]	$1+\frac{1}{T^2}-\frac{1}{T}$	$\frac{-2+T}{1-T+T^2}$
Knot[4, 1]	$-1+3\,T-T^2$	$\frac{T\,(-3+2\,T)}{1-3\,T+T^2}$
Knot[5, 1]	$\frac{1-T+T^2-T^3+T^4}{T^4}$	$\frac{-4+3\,T-2\,T^2+T^3}{1-T+T^2-T^3+T^4}$
Knot[5, 2]	$\frac{2-3\,T+2\,T^2}{T^3}$	$-\frac{2\,(3-3\,T+T^2)}{2-3\,T+2\,T^2}$
Knot[6, 1]	$5-\frac{2}{T}-2\,T$	$\frac{2\,(-1+T^2)}{(-2+T)\,(-1+2\,T)}$
Knot[6, 2]	$-3-\frac{1}{T^2}+\frac{3}{T}+3\,T-T^2$	$\frac{-2+3\,T-3\,T^3+2\,T^4}{1-3\,T+3\,T^2-3\,T^3+T^4}$
Knot[6, 3]	$5+\frac{1}{T^2}-\frac{3}{T}-3\,T+T^2$	$\frac{-2+3\,T-3\,T^3+2\,T^4}{1-3\,T+5\,T^2-3\,T^3+T^4}$
Knot[7, 1]	$\frac{1-T+T^2-T^3+T^4-T^5+T^6}{T^6}$	$\frac{-6+5\,T-4\,T^2+3\,T^3-2\,T^4+T^5}{1-T+T^2-T^3+T^4-T^5+T^6}$

Knot[7, 2]	$\frac{3-5\,T+3\,T^2}{T^4}$	$-\frac{3\,(4-5\,T+2\,T^2)}{3-5\,T+3\,T^2}$
Knot[7, 3]	$T^2\,(2-3\,T+3\,T^2-3\,T^3+2\,T^4)$	$\frac{4-9\,T+12\,T^2-15\,T^3+12\,T^4}{2-3\,T+3\,T^2-3\,T^3+2\,T^4}$
Knot[7, 4]	$T^3\,(4-7\,T+4\,T^2)$	$\frac{4\,(3-7\,T+5\,T^2)}{4-7\,T+4\,T^2}$
Knot[7, 5]	$\frac{2-4\,T+5\,T^2-4\,T^3+2\,T^4}{T^5}$	$\frac{-10+16\,T-15\,T^2+8\,T^3-2\,T^4}{2-4\,T+5\,T^2-4\,T^3+2\,T^4}$
Knot[7, 6]	$5-\frac{1}{T^3}+\frac{5}{T^2}-\frac{7}{T}-T$	$\frac{-3+10\,T-7\,T^2+T^4}{1-5\,T+7\,T^2-5\,T^3+T^4}$
Knot[7, 7]	$1-5\,T+9\,T^2-5\,T^3+T^4$	$\frac{T\,(-5+18\,T-15\,T^2+4\,T^3)}{1-5\,T+9\,T^2-5\,T^3+T^4}$
Knot[8, 1]	$-3-\frac{3}{T^2}+\frac{7}{T}$	$\frac{-6+7\,T}{3-7\,T+3\,T^2}$
Knot[8, 2]	$-\frac{1-3\,T+3\,T^2-3\,T^3+3\,T^4-3\,T^5+T^6}{T^4}$	$\frac{-4+9\,T-6\,T^2+3\,T^3-3\,T^5+2\,T^6}{1-3\,T+3\,T^2-3\,T^3+3\,T^4-3\,T^5+T^6}$
Knot[8, 3]	$-4+9\,T-4\,T^2$	$\frac{T\,(-9+8\,T)}{4-9\,T+4\,T^2}$
Knot[8, 4]	$5-\frac{2}{T}-5\,T+5\,T^2-2\,T^3$	$\frac{-2+5\,T^2-10\,T^3+6\,T^4}{2-5\,T+5\,T^2-5\,T^3+2\,T^4}$
Knot[8, 5]	$-(1-T+T^2)\,(1-2\,T+T^2-2\,T^3+T^4)$	$\frac{T\,(-3+8\,T-15\,T^2+16\,T^3-15\,T^4+6\,T^5)}{(1-T+T^2)\,(1-2\,T+T^2-2\,T^3+T^4)}$
Knot[8, 6]	$6-\frac{2}{T^3}+\frac{6}{T^2}-\frac{7}{T}-2\,T$	$\frac{-6+12\,T-7\,T^2+2\,T^4}{2-6\,T+7\,T^2-6\,T^3+2\,T^4}$
Knot[8, 7]	$5+\frac{1}{T^2}-\frac{3}{T}-5\,T+5\,T^2-3\,T^3+T^4$	$\frac{-2+3\,T-5\,T^3+10\,T^4-9\,T^5+4\,T^6}{1-3\,T+5\,T^2-5\,T^3+5\,T^4-3\,T^5+T^6}$
Knot[8, 8]	$-6+\frac{2}{T}+9\,T-6\,T^2+2\,T^3$	$\frac{-2+9\,T^2-12\,T^3+6\,T^4}{(2-2\,T+T^2)\,(1-2\,T+2\,T^2)}$
Knot[8, 9]	$-\frac{(-1+T-2\,T^2+T^3)\,(-1+2\,T-T^2+T^3)}{T^2}$	$\frac{-2+3\,T-7\,T^3+10\,T^4-9\,T^5+4\,T^6}{(-1+T-2\,T^2+T^3)\,(-1+2\,T-T^2+T^3)}$
Knot[8, 10]	$\frac{(1-T+T^2)^3}{T^2}$	$-\frac{2+T-4\,T^2}{1-T+T^2}$
Knot[8, 11]	$-\frac{(-2+T)\,(-1+2\,T)\,(1-T+T^2)}{T^3}$	$\frac{-6+14\,T-9\,T^2+2\,T^4}{2-7\,T+9\,T^2-7\,T^3+2\,T^4}$
Knot[8, 12]	$-7+\frac{1}{T}+13\,T-7\,T^2+T^3$	$\frac{-1+13\,T^2-14\,T^3+3\,T^4}{1-7\,T+13\,T^2-7\,T^3+T^4}$
Knot[8, 13]	$-7+\frac{2}{T}+11\,T-7\,T^2+2\,T^3$	$\frac{-2+11\,T^2-14\,T^3+6\,T^4}{2-7\,T+11\,T^2-7\,T^3+2\,T^4}$
Knot[8, 14]	$8-\frac{2}{T^3}+\frac{8}{T^2}-\frac{11}{T}-2\,T$	$\frac{-6+16\,T-11\,T^2+2\,T^4}{2-8\,T+11\,T^2-8\,T^3+2\,T^4}$
Knot[8, 15]	$\frac{(1-T+T^2)\,(3-5\,T+3\,T^2)}{T^5}$	$\frac{-15+32\,T-33\,T^2+16\,T^3-3\,T^4}{(1-T+T^2)\,(3-5\,T+3\,T^2)}$
Knot[8, 16]	$8+\frac{1}{T^4}-\frac{4}{T^3}+\frac{8}{T^2}-\frac{9}{T}-4\,T+T^2$	$\frac{-4+12\,T-16\,T^2+9\,T^3-4\,T^5+2\,T^6}{1-4\,T+8\,T^2-9\,T^3+8\,T^4-4\,T^5+T^6}$
Knot[8, 17]	$-8-\frac{1}{T^2}+\frac{4}{T}+11\,T-8\,T^2+4\,T^3-T^4$	$\frac{-2+4\,T-11\,T^3+16\,T^4-12\,T^5+4\,T^6}{1-4\,T+8\,T^2-11\,T^3+8\,T^4-4\,T^5+T^6}$
Knot[8, 18]	$-\frac{(1-3\,T+T^2)\,(1-T+T^2)^2}{T^3}$	$\frac{-3+7\,T-7\,T^3+3\,T^4}{(1-3\,T+T^2)\,(1-T+T^2)}$
Knot[8, 19]	$T\,(1-T+T^3-T^5+T^6)$	$\frac{1-2\,T+4\,T^3-6\,T^5+7\,T^6}{1-T+T^3-T^5+T^6}$
Knot[8, 20]	$\frac{(1-T+T^2)^2}{T^2}$	$\frac{2\,(-1+T^2)}{1-T+T^2}$
Knot[8, 21]	$-\frac{(1-3\,T+T^2)\,(1-T+T^2)}{T^3}$	$\frac{-3+8\,T-5\,T^2+T^4}{(1-3\,T+T^2)\,(1-T+T^2)}$